# Numeric series

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# Definitions

We will consider numeric series:

$$S = a_1 + a_2 + a_3 + \cdots + a_n + \ldots.$$
$$S = \sum_{n=1}^{\infty} a_n.$$

Both formulas must be considered as in pure formal sense. You should not try to find the sum very often the sum in general case does not exists!

# Example 1

$$
\begin{aligned}
S &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \ldots. \\
S &= 1 + \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} + \ldots\right), \\
S &= 1 + \frac{1}{2}S, \quad \left(1 - \frac{1}{2}\right)S = 1, \\
S &= \frac{1}{1 - \frac{1}{2}}, \quad S = 2.
\end{aligned}
$$

# Example 2

$$\begin{aligned} S &= 1 - 1 + 1 - 1 + 1 - 1 + \dots + (-1)^{n+1} + \dots, \\ S &= 1 - (1 - 1 + 1 - 1 + 1 - 1 + \dots + (-1)^{n} + \dots), \\ S &= 1 - S, \quad S = \frac{1}{2}. \end{aligned}$$

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots + \frac{(-1)^{n+1}}{n} + \dots.$$

Let's consider a finite sum:

$$S_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{(-1)^{n+1}}{n}$$

The rest term

$$R_n = -\frac{(-1)^{n+2}}{n+1}$$

# Examples

$$1+\frac{1}{3}+\frac{1}{5}-\frac{1}{2}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\frac{1}{15}-\frac{1}{4}+\cdots=$$
$$1.5(3)-0.5+0.4(884670)-0.25+\ldots.$$

# Definitions

The sum of several terms of the series:

$$S_N = \sum_{n=1}^{N-1} a_n$$

is called **partial sum.**

Let find the partial sum of the geometric series:

$$\begin{aligned} S_N &= \sum_{n=0}^{N-1} q^n = 1 + q + q^2 + \cdots + q^{N-1} \\ qS_N &= q + q^2 + \cdots + q^N, \\ S_N - qS_N &= 1 - q^n, \quad S_N = \frac{1-q^N}{1-q}. \end{aligned}$$

# Definitions

The series is called **convergent** if there exists the limit:

$$S = \lim_{N \to \infty} S_N.$$

If the limit does not exists. then the series is called **divergent series**.

# The convergence of the sequence

The Cauchy convergence test is a reformulated condition of convergence of the sequence $\{S_n\}$.

## Theorem

*The series converges if* $\forall \epsilon > 0 \, \exists N$ :

$$|\sum_{k=n}^{p} a_k| < \epsilon, \, \forall p > n > N.$$

**Proof** For convergent sequences one get the inequality $|S_p - S_n| < \epsilon, \; p > n > N$, then $|\sum_{k=n}^{p} a_k| < \epsilon, \forall p > n > N$.

# Geometric series

The sum of the geometric series:

$$S = \lim_{N \to \infty} \frac{1 - q^N}{1 - q} = \frac{1}{1 - q}.$$

The sum of the alternating series:

$$S = \sum_{n=0}^{\infty} (-1)^n, \quad S_{2N} = 0, \quad S_{2N+1} = 1$$

The alternating series

$$S = \sum_{n=0}^{\infty} (-1)^n$$

diverges.

# The necessary condition for convergence

## Theorem

*If the series*

$$S = \sum_{n=0}^{\infty} a_n$$

*converges, then*

$$\lim_{n\to\infty} a_n = 0.$$

**Proof.** Assume $\exists \lim_{n\to\infty} S_n$, $S_{n+1} = S_n + a_n$. Due to the uniqueness of the limit
$\forall \epsilon > 0 \, \exists N, \forall n > N : |S_n - S_{n-1}| < \epsilon \Rightarrow |a_n| < \epsilon$.

# Divergence test

## Theorem

*If*

$$\lim_{n\to\infty} a_n \neq 0.$$

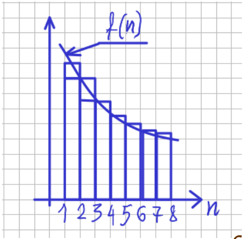
*then the series*

$$S = \sum_{n=0}^{\infty} a_n$$

*diverges.*

**Proof** of this theorem very close to the proof of the previous theorem.

# The integral test theorem




Let $f(x)$ is positive continuous decreasing function, then $\int_1^\infty f(x)dx$ and $\sum_{n=1}^\infty f(n)$ either both converge or diverge.

**Proof.**

$$f(n) \geq \int_n^{n+1} f(x)dx \geq f(n+1)$$

$$\sum_{n=1}^{N} f(n) \geq \int_1^{N+1} f(x)dx \geq \sum_{n=1}^{N} f(n+1)$$

$$S_n \geq \int_1^{N+1} f(x)dx \geq S_{N+1} - f(1).$$

# The integral test theorem. Proof

If the integral diverges, then $\{S_n\}$ diverges. If the integral converges then $\{S_{N+1}\}$ converges and vice verge.

# Harmonic series

$$S = \sum_{n=1}^{\infty} \frac{1}{n} \text{ related to } \int_1^N \frac{dx}{x} = \log(N).$$

The integral diverge and due to the integral test the harmonic series diverges.

# Slowly convergent and divergent series

$$S = \sum_{n=1}^{\infty} \frac{1}{n \log^2(n)},$$

$$I(N) = \int_2^N \frac{dx}{x \log^2(x)} = \frac{1}{\log(2)} - \frac{1}{\log(N)}.$$

To obtain right value of third decimal digit one must sum over of $10^{300}$ terms.

# Slowly divergent series

$$S = \sum_{n=1}^{\infty} \frac{1}{n \log(n)},$$

$$I(N) = \int_2^N \frac{dx}{x \log(x)} = \log(\log(N)) - \log(\log(2)).$$

The series diverges, but too slowly

$$S_{10^6} \sim 2.625791914476011, \quad S_{10^{12}} \sim 3.318939095035956.$$

# Absolute and conditional convergence

The series

$$S = \sum_{n=0}^{\infty} a_n$$

converges **absolute** if the series

$$s = \sum_{n=0}^{\infty} |a_n|$$

converges.

# Ratio test theorem

If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = q < 1,$$

then the series converges absolute. If $q > 1$ then the series diverges.

**Proof.** $\forall\epsilon > 0\,\exists N, \forall m > N, \exists Q : 1 > Q > q$

$$|a_{m+k}| < |a_{m+k-1}|Q < \cdots < |a_m|Q^k,$$

$$\sum_{k=m}^{\infty}|a_m|Q^k = \frac{|a_m|}{1-Q}.$$

$$\sum_{n=1}^{\infty}|a_n| < \sum_{n=1}^{N}|a_n| + \frac{|a_{N+1}|}{1-Q}.$$

The series converges.

# Ratio test theorem. Proof

If $q > 1$, then $\forall \epsilon > 0 \, \exists N, \forall m > N, \exists Q : 1 < Q < q$

$$\sum_{k=m}^{M} |a_m| Q^k \to \infty$$

as $M \to \infty$, therefore the series diverges.

# Ratio test. An example

$$S = \sum_{n=0}^{\infty} \frac{n^2}{3^n},$$

$$\lim_{n\to\infty} \frac{(n+1)^2}{3^{n+1}} \frac{3^n}{n^2} = \frac{1}{3} \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = \frac{1}{3}.$$

The series converges.

# Root test theorem

If

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = q < 1,$$

then the series

$$s = \sum_{n=0}^{\infty} |a_n|$$

converges. If

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = q > 1,$$

the series diverges.

# Root test theorem. Proof

$\forall \epsilon > 0 \, \exists N, \forall m > N, \exists Q : 1 > Q > q, \ \sqrt[m]{|a_m|} < Q, \quad |a_m| < Q^m.$

$$s = \sum_{n=0}^{\infty} |a_n| \leq \sum_{n=0}^{N} |a_n| + \sum_{n=N+1}^{\infty} Q^n$$

The series converges.

# Root test theorem. Proof

If

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = q > 1,$$

$\forall \epsilon > 0 \, \exists N, \forall m > N, \exists Q : 1 < Q < q,$
$\sqrt[m]{|a_m|} > Q, \quad |a_m| > Q^m.$

$$s = \sum_{n=0}^{\infty} |a_n| \geq \sum_{n=0}^{N} |a_n| + \sum_{n=N+1}^{\infty} Q^n$$

$$\sum_{n=N+1}^{M} Q^n \to \infty, \quad M \to \infty.$$

The series diverges.

# Root test. An example

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{n}{3^n}, \\ \lim_{n\to\infty} \sqrt{\frac{n}{3^n}} &= \frac{1}{3} \lim_{n\to\infty} \sqrt[n]{n} \\ \lim_{n\to\infty} \log(\sqrt[n]{n}) &= \lim_{n\to\infty} \frac{\log(n)}{n} = 0, \\ \frac{1}{3} \lim_{n\to\infty} \sqrt[n]{n} &= \frac{1}{3}. \end{aligned}$$

The series converges.

# Comparison test

If $\exists N$ such that $\forall n > N$ the inequalities $a_n > b_n > 0$ are true and the series

$$S = \sum_{n=1}^{\infty} a_n$$

converges, then the series

$$s = \sum_{n=1}^{\infty} b_n$$

converges.

# Comparison test. Proof

**Proof.**

$$s = \sum_{n=1}^{N} b_n + \sum_{n=N+1}^{\infty} b_n.$$

The first sun is bounded due to the bounded number of the terms and the second series is bounded due to the properties of $a_n > b_n > 0$ and the convergence of the series $S$.

# Leibniz theorem about alternating series

Let $u_n > u_{n+1}$ then

$$S = \sum_{n=0}^{\infty} (-1)^n u_n$$

converges.

**Proof.**

$$S = (u_0 - u_1) + (u_2 - u_3) + (u_4 - u_5) + \ldots.$$

Define $\sigma_n = (u_{2n} - u_{2n+1}) > 0$, then the partial sums

$$S_n = \sum_{n=0}^{N} \sigma_n$$

increase.

# Leibniz theorem about alternating series. Proof

$$S = u_0 - (u_1 - u_2) - (u_3 - u_4) - (u_5 - u_6) - u_7 + \dots.$$

Define $\tau_n = u_{2n-1} - u_{2n} > 0$. The partial sums

$$S_n = u_0 - \tau_1 - \tau_2 - \dots$$

Then $S_n$ is bounded. Therefore the sequence $S_n$ has a limit. The alternating series converges.

# Alternating series. An example

The series

$$S = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

converges.

# Riemann series theorem

Consider the series

$$S = \sum_{n=0}^{\infty} (-1)^n u_n,$$

where $u_n > u_{n+1} > 0$, $u_n \to 0$ as $n \to \infty$ and both series

$$s_+ = \sum_{n=0}^{\infty} u_{2n}, \quad s_- = \sum_{n=0}^{\infty} u_{2n+1}$$

diverge.

Then one can rearrangement of the series such way, that the sum might be any number.

# Summary