Sapienti sat-1

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Series. Definition of convergence

A definition of series as symbol:

$$S = \sum_{n=1}^{\infty} a_n.$$

The sum of several terms of the series:

$$S_N = \sum_{n=1}^{N} a_n$$

is called **partial sum.**
The series is called **convergent** if the limit exists:

$$S = \lim_{N\to\infty} S_N.$$

Diversity of convergence. Chesaro summation

Divergent series:

$$S = \sum_{N=1}^{\infty}(-1)^{n-1}.$$

A definition of convergence by Chezaro:

$$S = \lim_{n\to\infty}\frac{1}{n}\sum_{n+1}^{n} s_n, \quad s_n = \sum_{k=1}^{n} a_k.$$

In this case the Chesaro sum is follows:

$$\sum_{n=1}^{\infty}(-1)^{n-1} = \frac{1}{2}.$$

Diversity of convergence. Borel summation

A following series diverges at any point x:

$$\phi(x) \sim \sum_{n=1}^{\infty}(-1)^{n-1}x^n n!.$$

Let's define the Borel summation:

$$S = \int_0^{\infty} e^{-t}\sum_{n=0}^{\infty}\frac{t^n}{n!}a_n, \quad A(z) = \sum_{k=0}^{\infty} a_k z^k.$$

$$\int_0^{\infty} e^{-t}\sum_{n=0}^{\infty}(-1)^n\frac{t^n}{n!}n!(-1)^n z^n dt =$$

$$\int_0^{\infty} e^{-t}\sum_{n=0}^{\infty}(tz)^n dt = \int_0^{\infty}\frac{e^{-t}dt}{1+tz}.$$

Riemann series theorem

Consider the series

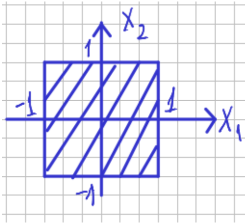
$$S = \sum_{n=0}^{\infty} (-1)^n u_n,$$

where $u_n > u_{n+1} > 0$, $u_n \to 0$ as $n \to \infty$ and both series

$$s_+ = \sum_{n=0}^{\infty} u_{2n}, \quad s_- = \sum_{n=0}^{\infty} u_{2n+1}$$

diverge.
Then one can rearrangement of the series such way, that the sum might be any number.

Chebyshev distance

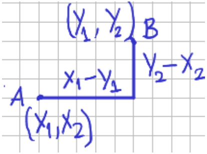


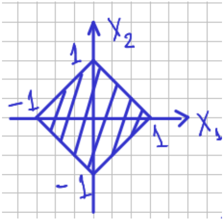
If an element is
defined by two properties of the different nature, then to define the difference two objects $A(x_1, x_2)$ and $B(y_1, y_2)$ one can consider a lot of variants.
One can define the distance $\rho(A, B)$ as **Chebyshev distance**:

$$\rho_C(A, B) = \max_{i=1,2} |x_i - y_i|.$$

The ball with radius 1 in the term of Chebyshev distance.

Manhattan distance



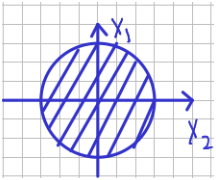


Another example of the distance might be defined as **Manhattan distance**:

$$\rho_M(A, B) = |x_1 - y_1| + |x_2 - y_2|.$$

The ball with radius
1 in terms of the Manhattan distance.

Euclidean distance

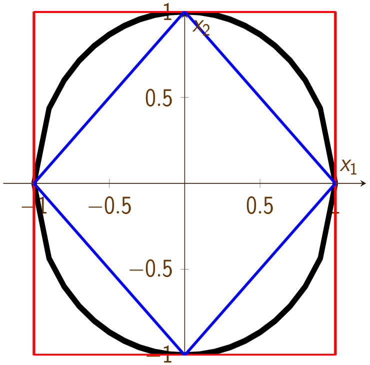


At the end
one can define as **Euclidean distance**:

$$\rho_E(A, B) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

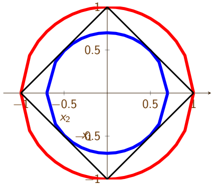
The ball with
radius 1 in terms of Euclidean distance.

The two-dimensional unit ball



The red lines are border of of the ball of radius $R = 1$ for **Chebyshev distance.** The blue lines define the border of the ball of the radius $R = 1$ for **the Manhattan distance.** The **black line** is the border of the of the ball of the radius $R = 1$ for **the Euclidean distance.**

Properties of the distance



The defined
distance has the following properties.

- $\rho(A, B) = 0 \Leftrightarrow A \equiv B$;
- $\rho(A, B) \geq 0$;
- $\rho(A, B) + \rho(B, C) \geq \rho(A, C)$.

The picture shows that Manhattan norm is equivalent to Euclidean one.

Theorem about equivalence of norms

$\forall || \cdot ||_{1,2},\ \exists C_1, C_2 > 0:$

$$C_1||X||_1 \leq ||X||_2 \leq C_2||X||_1.$$

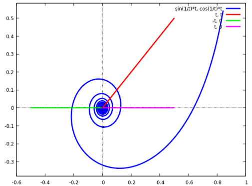
A limit of a function of two variables

If $\forall \epsilon > 0 \; \exists \delta(\epsilon) : |F(Y) - A| < \epsilon \; \forall Y : \; ||Y - X|| < \delta$, then the value A is a limit of the function F in the point X:

$$\lim_{||Y-X|| \to 0} f(X) = A.$$

A function which has a limit in all pints of given set is continuous function on this set.

Examples of the approaches to the given point

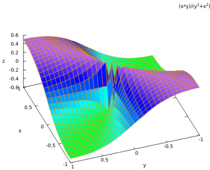


Let's consider
the origin as a point of approaching.

$$X \to 0, \Rightarrow ||X|| \to 0 :$$

1. $X = (x_1, x_2),\ ||X|| \to 0\ \Rightarrow\ x_1 \to 0,\ x_2 \to 0;$
2. $X = (x_1, x_2),\ ||X|| \to 0 \Rightarrow x_2 \to 0,\ x_1 \to 0;$
3. $X = (x_1, x_2),\ ||X|| \to 0 \Rightarrow x_1 = kx_2 \to 0,\ x_2 \to 0;$
4. $X = (x_1, x_2),\ ||X|| \to 0 \Rightarrow x_1 = r\cos(\alpha), x_2 = r\sin(\alpha), r \to 0;$
5. $X = (x_1, x_2),\ ||X|| \to 0 \Rightarrow x_1 = r\cos(\alpha/r), x_2 = r\sin(\alpha/r), r \to 0.$

Examples of the limits



$$\lim_{X\to 0}(x_1^2+x_2^2)=0,$$

$$\lim_{X\to(2,1)}(x_1^2+x_2^2)=4+1=5;$$

$$\lim_{x_1\to 0}\lim_{x_2\to 0}\frac{x_1x_2}{x_1^2+x_2^2}=\lim_{x_1\to 0}0=0;$$

$$\lim_{x_1\to 0}\left.\frac{x_1x_2}{x_1^2+x_2^2}\right|_{x_2=kx_1}=\lim_{x_1\to 0}\frac{x_1kx_1}{x_1^2+k^2x_1^2}=\frac{k}{1+k^2},$$

$$\lim_{r\to 0}\left.\frac{x_1x_2}{x_1^2+x_2^2}\right|_{\substack{x_1=r\cos(\alpha),\\ x_2=r\sin(\alpha)}}=\lim_{r\to 0}\frac{r^2\cos(\alpha)\sin(\alpha))}{r^2}=\frac{1}{2}\sin(2\alpha).$$

Iterated limits and limit interchanging

Let's consider the iterated limits adna limit as $||x|| \to 0$:

$$\lim_{x_1\to 0}\lim_{x_2\to 0}\frac{x_1}{x_1+x_2} = \lim_{x_1\to 0}\frac{x_1}{x_1} = 1,$$

$$\lim_{x_2\to 0}\lim x_1 \to 0\frac{x_1}{x_1+x_2} = \lim_{x_2\to 0} 0 = 0.$$

$$\lim_{r\to 0}\frac{x_1}{x_1+x_2} = \lim_{r\to 0}\frac{r\cos(\phi)}{r\cos(\phi)+r\sin(\phi)} = \frac{\cos(\phi)}{\cos(\phi)+\sin(\phi)}.$$

One can see both iterated limits exist but they are different and a limit as $||x|| \to 0$ does not exists.
This examples show that the changing of the iterated limits can change the answer.
The question is: When can be changed the iterated limits?

The theorem about interchanging the iterated limits

If $\exists \lim_{X\to 0} = A$ and $\exists \lim_{x_1\to 0} f(x_1, x_2) = f(0, x_2) \forall x_2 \neq 0$, then

$$\lim_{x_1\to 0}\lim_{x_2\to 0} f(x_1, x_2) = \lim_{x_2\to 0}\lim_{x_1\to 0} f(x_1, x_2) = A.$$

Proof.

$$||X|| < \delta(\epsilon) \Rightarrow |f(x_1, x_2) - A| < \epsilon \ \Rightarrow |f(x_1, 0) - A| < \epsilon, \Rightarrow \lim_{x_1\to 0} = A, \Rightarrow \lim_{x_1\to 0}\lim_{x_2\to 0} f(x_1, x_2) = A.$$

Invariant form of the differential

Consider the changing of coordinates for x, y:

$$x = x(u, v),\ \ y = y(u, v),$$

Below we suppose that the functions $x(u, v)$ and $y(u, v)$ are differentiable.

$$\begin{aligned}
df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \\
&= \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right) + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right) = \\
&= \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\right)du + \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\right)dv \\
&= \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv.
\end{aligned}$$

Differential for the N-dimensional function

The changing of the variables in general form looks like:

$$X = X(U), \quad X(U) = (x_1(u_1, \ldots, u_n), \ldots, x_N(u_1, \ldots, u_N)).$$

In this case the differential has the same form:

$$df = \sum_{k=1}^{N} \frac{\partial f}{\partial x_k} dx_k = \sum_{k=1}^{N} \frac{\partial f}{\partial u_k} du_k.$$

As well as the differential is the primary (linear) part of the function changing then the vector

$$\vec{S} = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{ptx_N} \right)$$

defines the direction of the grows of the function for the given point X.

Gradient of the function

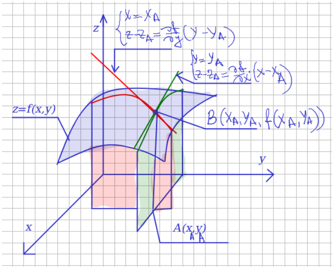
The vector $\vec{v} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ is called gradient of the function $f(x, y)$ at the point (x, y). The gradient can be written by following equivalent definitions:

$$\vec{\text{grad}}(f) \equiv \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right);$$
$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

Define the differential of the independent variables as $d\vec{X} = (dx_1, dx_2, \ldots, dx_N)$. The differential of the function can be written as scalar product:

$$(\vec{\nabla} f, d\vec{X}) = \sum_{k=1}^{N} \frac{\partial f}{\partial x_k} dx_k.$$

Geometrical sense of the partial derivatives



Let us consider surface $z = f(x, y)$. Consider a dissection of the surface by the plain $y = y_0$, $y_0 = \mathrm{const}$. The intersection of the surface and plain defines the curve one-dimensional curve $z = f(x, y_0)$ and the angle of the tangent line for the curve at the point x_0 is $\frac{\partial f}{\partial x}$ The same for the curve $z = f(x_0, y)$ one gets the angle of the tangent line for the curve $z = f(x_0, y)$ is $\frac{\partial f}{\partial y}$.

A normal vector

Rewrite the equation for the surface in the form:

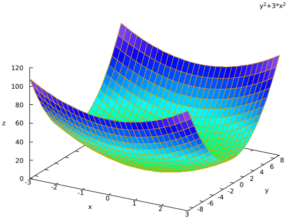
$$z - f(x, y) = 0.$$

The differential on the surface should be following:

$$dz - \frac{\partial f}{\partial x} dx - \frac{\partial f}{\partial y} dy = 0.$$

This equality should be fulfill for any curve on this surface $(x(t), y(t), z(t))$, then these equality is the scalar product for the vector $\vec{N} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right)$ and the vector of differential for any curve on the surface. As well as the differential defines the tangent lines for the surface, then $\vec{N}$ is a normal vector for the surface at the point $(x_0, y_0, f(x_0, y_0))$.

Definition of extreme point

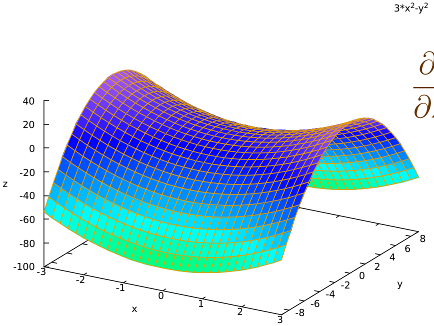


The point
of differentiable function $f(X)$
where all derivatives of first order
are zero is called extreme point.
For following functions point
$A=(0,0)$ is an extreme point:

$$f(x_1,x_2)=3x_1^2+x_2^2,$$
$$\frac{\partial f}{\partial x_1}=6x_1,\ \frac{\partial f}{\partial x_1}=2x_2;$$

$$f(x_1,x_2)=-3x_1^2-x_2^2,\quad \frac{\partial f}{\partial x_1}=-6x_1,\ \frac{\partial f}{\partial x_1}=-2x_2;$$

Saddle point



$$f(x_1, x_2) = 3x_1^2 - x_2^2,$$

$$\frac{\partial f}{\partial x_1} = 6x_1, \ \ \frac{\partial f}{\partial x_1} = -2x_2.$$

Necessary conditions for the extreme points

Theorem

If $f(X)$ is differentiable, then $\frac{\partial f}{\partial x_k} = 0$, $\forall k \in \{1, \ldots, N\}$ at the interior maxima or minima point.

Theorem

Let $f(X)$ be differentiable function and all derivatives of the first and second order are continuous, then

$$\frac{\partial^2 f}{\partial x_k \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_k}.$$

Sufficient conditions for minima and maxima

Theorem

Let $f(X)$ be twice differentiable function of two variables and the point $A = (0, 0)$ is extreme point

- *if* $\left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right)^2 - \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} < 0$ *and* $\frac{\partial^2 f}{\partial x_1^2} < 0$*, then A is a maxima;*
- *if* $\left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right)^2 - \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} < 0$ *and* $\frac{\partial^2 f}{\partial x_1^2} > 0$ *then A is a minima;*
- *if* $\left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right)^2 - \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} > 0$*, then A is a saddle point;*
- *if* $\left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right)^2 - \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} = 0$*, then we need an additional studies.*

Optimal problems with constrains

Let us consider the level of f the function should touch to the plain. Then the gradients of f and the constraint curve are collinear:

$$\vec{\nabla} f = -\lambda \vec{\nabla} \phi.$$

Additional condition is the constrain $\phi(x, y, z) \equiv ax + by + cz + d = 0$.

$$\phi(x, y, z) = 0.$$

Define the Lagrange function (Lagrangian):

$$L(x, y, z) = f(x, y, z) + \lambda \phi(x, y, z).$$

Lagrange multipliers. General case

$f = f(X)$ and constraints $\phi_k(X)$, $k = 1, \dots, m$, then:

$$L = f(x) + \sum_{k=1}^{m} \lambda_k \phi_k(X).$$

The necessary condition for the extreme point:

$$\vec{\nabla} L(X, \Lambda) = 0, \;\; \Lambda = (\lambda_1, \dots, \lambda_m).$$

The fastest gradient-wise descent

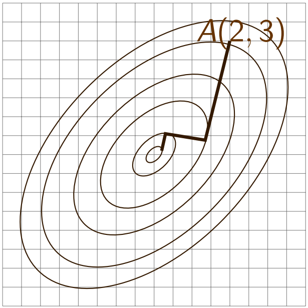
The gradient descent method for the opposite gradient direction as function of the variable Δ:

$$\Phi(\Delta) = F(\vec{x} - \vec{\nabla} F(\vec{x}) \cdot \Delta), \quad \{x_1, \ldots, x_N\} = \mathrm{const}\,.$$

So we seek the minimum of the one dimension function $\Phi(\Delta)$ on the given direction.

1. Define an interval $\Delta \in [0, b]$ such that $\Phi(\Delta) \leq \Phi(0)$.
2. Find a minimum $\Phi(\Delta^*)$ on $\Delta \in [0, b]$ using for example a bisection method.
3. The point $\vec{X} = \vec{x} - \vec{\nabla} F(\vec{x}) \cdot \Delta^*$ is considered as next position for the next step.
4. If $||\vec{X} - \vec{x}|| > \delta$, then this process repeats.

An example



Consider the fastest gradient descent for the function

$$f(x_1, x_2) = (x_1+x_2)^2+3(x_1-x_2)^2.$$

The level curves are ellipses with big semi axis along the straight line $x_1 = x_2$ and the minimum is $(0,0)$.

Primary concepts for definition of a differentiable manifold

- $a_1x_1 + a_2x_2 + c = 0$, is a one function of one variable: $x_1(x_2) = -\frac{1}{a_1}(a_2x_2 + c)$ or $x_2(x_1) = -\frac{1}{a_2}(a_1x_1 + c)$. Both forms are appropriated if $a_{1,2} \neq 0$.
- $a_1x_1 + a_2x_2 + a_3x_3 + c = 0$ is a a function of two variables: $x_k = -\frac{1}{a_k}\left(\sum_{n\neq k} a_nx_n + c\right)$.
- The following two equals define one dimensional function.

$$a_1x_1 + a_2x_2 + a_3x_3 + c = 0,\ \ b_1x_1 + b_2x_2 + b_3x_3 + d = 0.$$

- In a general case the m equalities of N variables define $N - m$ dimensional implicit function:

$$f_k(x_1, \ldots, x_N) = 0,\ k = 1, \ldots, m.$$

Primary concepts for definition of differentiable manifolds

- Does the formula $\sum_{k=1}^{3} x_k^2 - R^2 = 0$ define a two-dimensional function?

$$x_i = +\sqrt{R^2 - \sum_{k=1, k\neq i}^{3} x_k^2}, \; x_i = -\sqrt{R^2 - \sum_{k=1, k\neq i}^{3} x_k^2}.$$

The answer looks like NO! because one obtains two different values for x_i for the same set of coordinates $\{x_k\}_{i\neq k}$.

Primary concepts for definition of differentiable manifolds

- Another point of view for the spherical coordinates:

  $x_1 = R\sin(\theta)\cos(\phi),\ x_2 = R\sin(\theta)\sin(\phi),\ x_3 = R\cos(\theta).$

  So we can see that in the spherical coordinate system one obtain one-to-one map $[0,\pi]\times[0,2\pi) \to$ a set of x_1, x_2, x_3.

We need a generalization for the function definition.

Theorem about explicit form of the function

Let's consider

$$f_k(x_1, \ldots, x_n) = 0, k \in \{1, \ldots, M\},\ n \in \{1, \ldots, N\},$$

where all f_k are continuously differentiable functions at the origin.
If a rank of the matrix

$$S = \left(\frac{\partial f_k}{\partial x_n}\right),\ s_{k,n} = \frac{\partial f_k}{\partial x_n}$$

is equal M at the origin then exists a neighborhood ofthe origin, where the implicit function can be rewritten in an explicit form as a function of $N - M$ independent variables.

A definition of differentiable manifold

The set in N dimensional space is called $N-M$ dimensional differentiable manifold if for any point A of the set exists neighborhood of the A such that $\exists \epsilon > 0$ and the manifold can be defined by

$$x_k = F_k(x_1, \ldots, x_{N-M}) = 0,\ k \in \{M, \ldots, N\},\ \forall x : ||x|| < \epsilon.$$

The set of maps covered all range of the variables is called an atlas.

Definition for a Jacobian

Let's consider the changing of variables:

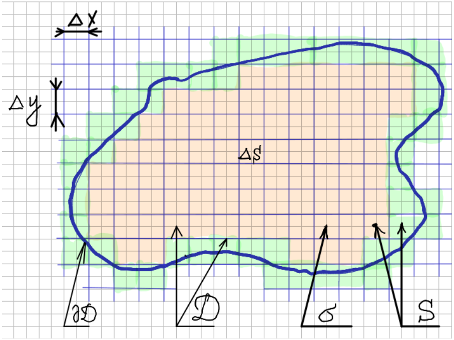
$$y_k = f_k(x).$$

the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ & \dots & \\ \frac{\partial f_k}{\partial x_1} & \dots & \frac{\partial f_k}{\partial x_n} \end{pmatrix}$$

is called Jacobian.

External integral sum

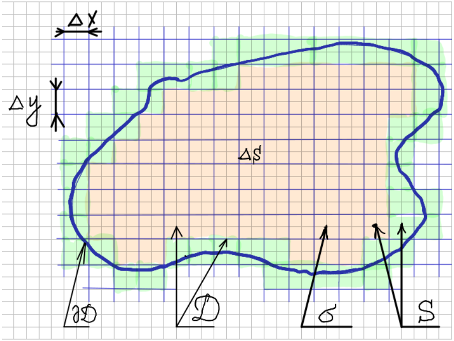


Consider an area $\mathcal{D}$ on the plane. Divide the area on a mesh with steps Δx and Δy. rectangle element of the plane $\Delta s = \Delta x \Delta y$. Cover the $\mathcal{D}$ by the rectangles $\Delta s = \Delta x \Delta y$ the and define the sum of the rectangles, which cover the $\mathcal{D}$:

$$S = \sum_{N} \Delta s.$$

Here N is the number of the elements Δs which covered the area $\mathcal{D}$.

Internal integral sum



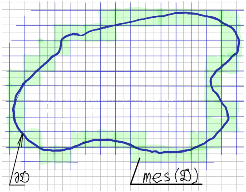
Define σ as a sum of the rectangles Δs which are internal of the $\mathcal{D}$:

$$\sigma = \sum_M \Delta s.$$

Here M is number of the internal rectangles for the $\mathcal{D}$, $M \leq N$ Then the area of the figure $\mathcal{D}$:

$$\sum_M \Delta s \leq \mathrm{mes}\mathcal{D} \leq \sum_N \Delta s.$$

An area of the border



Define a difference between sum external and internal rectangles as a area of the border:

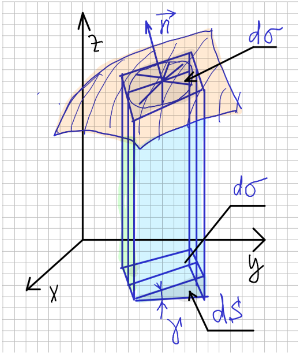
$$\mathrm{mes}(\partial\mathcal{D}) \leq (N - M)\Delta s.$$

Theorem *A measure of a rectifiable curve is equal to zero.*

Theorem. *If a border $\partial\mathcal{D}$ of a certain area $\mathcal{D}$ is rectifiable curve, then the area is measurable.*

example. *Koch's snowflake.*

Geometrical sense of the double integral



The projection of $d\sigma$ and dS connected by a formula:

$$d\sigma \cdot \cos(\gamma) = dS \Rightarrow d\sigma = \frac{dS}{\cos(\gamma)},$$

define $\vec{e}_3 = (0, 0, 1)$, then

$$\cos(\gamma) = (\vec{n}, \vec{e}_3) = \frac{1}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}}.$$

$$\sigma = \iint_{\mathcal{D}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} ds.$$

Properties of the double integral

The sum of the integrals. Consider continuous function $f(x,y)$ on $\mathcal{D}$. Let $\mathrm{mes}(\partial\mathcal{D})$, $\mathcal{D}=\mathcal{D}_1\cup\mathcal{D}_2$, $\mathcal{D}_1\cap\mathcal{D}_2=0$ and $\mathrm{mes}(\partial\mathcal{D}_{1,2})=0$, then

$$\int_{\mathcal{D}} f(x,y)ds=\int_{\mathcal{D}_1} f(x,y)ds+\int_{\mathcal{D}_2} f(x,y)ds.$$

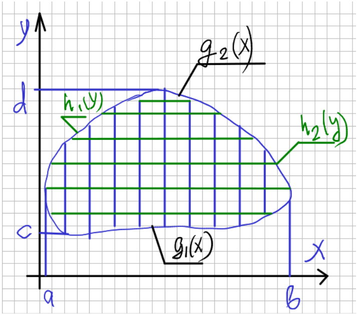
Estimation of the double integral. Let $f_m=\min_{(x,y)\in\mathcal{D}} f(x,y)$, $F_m=\max_{(x,y)\in\mathcal{D}} f(x,y)$, then

$$S\,f_m\le\int_{\mathcal{D}} f(x,y)ds\le S\,F_m$$

The mean value of the double integral. There exists (x_m,y_m) such that:

$$\int_{\mathcal{D}} f(x,y)ds=Sf(x_m,y_m).$$

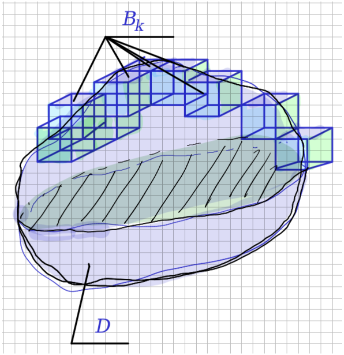
Fubini's theorem



Let $f(x,y)$ be a continuous function defined on region $\mathcal{D}$:
$\mathcal{D} = [a,b] \times [g_1(x), g_2(x)]$
or, the same,
$\mathcal{D} = [h_1(y), h_2(y)] \times [c,d]$
where
$g_{12}(x)$ and $h_{12}(y)$ continuous functions in the xy-plane.
Then the double integral of f over $\mathcal{D}$ can be expressed as an iterated integral:

$$\iint_{\mathcal{D}} f(x,y)ds = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

Definition of measure for volume of 3D body



A subset $\mathcal{D}$ of $\mathbb{R}^3$ has a measurable 3D measure if there exists a non-negative real number V such that $\forall \epsilon > 0$, $\exists \cup_{k=1}^n B_k$ of rectangular boxes B_k such that $\mathcal{D} \subset \bigcup_{i=1}^n B_i$ and

$$\sum_{i=1}^n |B_i| < V + \epsilon,$$

where $|B_i|$ denotes the volume of the rectangular box B_i. The number V is called the 3D volume of $\mathcal{D}$, denoted by $vol(\mathcal{D})$.

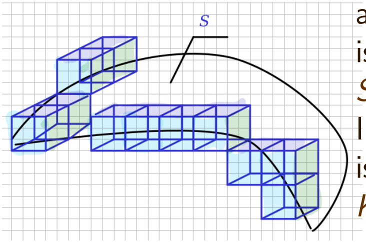
A Darboux criteria for the existence of the measure for given 3D body

A subset $\mathcal{D}$ of $\mathbb{R}^3$ is measurable if and only if $\forall\epsilon > 0$, $\exists A = \cup_{k=1}^{n} A_k$ and $B = \cup_{k=1}^{n} B_k$ where A_k and B_k are rectangular boxes, $A \subset \mathcal{D} \subset B$ and

$$|B \setminus A| < \epsilon,$$

where $|B \setminus A|$ denotes the 3D volume of the set difference $B \setminus A$.
In other words, a subset $\mathcal{D}$ of $\mathbb{R}^3$ is measurable if and only if it can be enclosed by two finite unions of rectangular boxes with arbitrarily close volumes.

A theorem about measurable of bounded 3d body with measurable border surface



Let $S = \partial\mathcal{D}$ be a smooth and bounded surface in $\mathbb{R}^3$. If S is measurable as 2D surface, then S has zero volume (3D measure). In other words, if S is a smooth 2D surface in compact $K \in \mathbb{R}^3$, then its area is zero.

Counterexample. The Mundelbulb 3D fractal

Stolen from Internet!

Definition of triple integral

Let D be a bounded and measurable domain in $\mathbb{R}^3$ and let $f : D \to \mathbb{R}$ be a function. The **triple integral** of f over D is denoted by

$$\iiint_D f(x, y, z)\, dx\, dy\, dz,$$

and is defined as the limit of Riemann sums as the mesh size approaches zero:

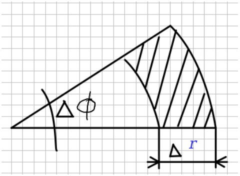
$$\iiint\limits_D f(x, y, z)\, dx\, dy\, dz = \lim_{\max(\Delta V_{ijk}) \to 0} \sum_{i,j,k} f(\xi_{ijk}, \eta_{ijk}, \zeta_{ijk})\, \Delta V_{ijk},$$

where $(\xi_{ijk}, \eta_{ijk}, \zeta_{ijk}) \in V_{ijk}$, and $\Delta V_{ijk} = vol(V_{ijk})$.

Theorem about existence of triple integrals

(Existence of Triple Integral) Let D be a bounded and measurable domain in $\mathbb{R}^3$, and let $f : D \to \mathbb{R}$ be a function. If f is continuous on D, then the triple integral $\iiint\limits_D f(x, y, z)\, dx\, dy\, dz$ exists.

From Cartesian to polar coordinate system



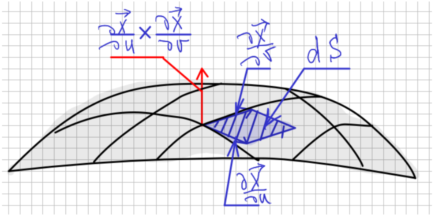
Consider
an elementary plate of the area
on the plane in a polar coordinates.

$$ds = r\,dr\,d\phi.$$

Consider an integral over an area with rectifiable border:

$$\iint_{\mathcal{D}} dxdy = \iint_{\mathcal{D}} ds = \iint_{\mathcal{D}} rdrd\phi$$

Changing of variables in two dimensional integrals



Let's consider a smooth surface S defined by a parametrization $\vec{\mathbf{x}} = \vec{\mathbf{x}}(u, v)$, where (u, v) are parameters in some domain $D \subset \mathbb{R}^2$. The elementary area of S at a point $\vec{\mathbf{x}}(u_0, v_0)$ is given by:

$$dS = \|\frac{\partial \vec{\mathbf{x}}}{\partial u} \times \frac{\partial \vec{\mathbf{x}}}{\partial v}\| du\, dv$$

where $\|\cdot\|$ denotes the Euclidean norm, and $\frac{\partial \vec{\mathbf{x}}}{\partial u}$, $\frac{\partial \vec{\mathbf{x}}}{\partial v}$ are the partial derivatives of $\vec{\mathbf{x}}$ with respect to u and v, respectively.

Changing of variables in two dimensional integrals

Now, suppose we have a change of variables $(u, v) = (u(r, s), v(r, s))$. Let $\vec{\mathbf{y}} = \vec{\mathbf{x}}(u(r, s), v(r, s))$ be a new parametrization of the surface S in terms of the new variables (r, s). Then the partial derivatives of $\vec{\mathbf{y}}$ with respect to r and s are given by the chain rule:

$$\frac{\partial \vec{\mathbf{y}}}{\partial r} = \frac{\partial \vec{\mathbf{x}}}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial \vec{\mathbf{x}}}{\partial v}\frac{\partial v}{\partial r}$$

and

$$\frac{\partial \vec{\mathbf{y}}}{\partial s} = \frac{\partial \vec{\mathbf{x}}}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial \vec{\mathbf{x}}}{\partial v}\frac{\partial v}{\partial s}$$

Changing of variables in two dimensional integrals

Taking the cross product of these vectors, we have:

$$\frac{\partial \vec{\mathbf{y}}}{\partial r} \times \frac{\partial \vec{\mathbf{y}}}{\partial s} = \left(\frac{\partial u}{\partial r}\frac{\partial v}{\partial s} - \frac{\partial u}{\partial s}\frac{\partial v}{\partial r}\right)\frac{\partial \vec{\mathbf{x}}}{\partial u} \times \frac{\partial \vec{\mathbf{x}}}{\partial v}$$

where we have used the fact that $\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}$ is a constant vector on the surface S.

Changing of variables in two dimensional integrals

Therefore, the new elementary area dS' in terms of the variables (r, s) is given by:

$$dS' = \|\frac{\partial \vec{\mathbf{y}}}{\partial r} \times \frac{\partial \vec{\mathbf{y}}}{\partial s}\| dr\, ds = \left|\frac{\partial u}{\partial r}\frac{\partial v}{\partial s} - \frac{\partial u}{\partial s}\frac{\partial v}{\partial r}\right| dS$$

where we have used the fact that the cross product of two vectors has the same Euclidean norm as their determinant.

Changing of variables in two dimensional integrals

Therefore, the elementary area changes by a factor of $\left|\frac{\partial u}{\partial r}\frac{\partial v}{\partial s} - \frac{\partial u}{\partial s}\frac{\partial v}{\partial r}\right|$ when changing variables from (u, v) to (r, s).
This is known as the Jacobian determinant of the change of variables, and it appears in many areas of mathematics, including multivariable calculus and differential geometry.

Changing of variables in triple integrals

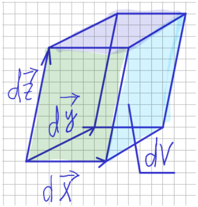
Consider a function $f(x, y, z)$ defined on a region D in three-dimensional space, and express the integral of f over D in terms of a new set of coordinates (u, v, w), where $x = x(u, v, w)$, $y = y(u, v, w)$, and $z = z(u, v, w)$. Then the triple integral can be written as:

$$\iiint_D f(x, y, z)dV =$$

$$\iiint_{D'} f(x(u, v, w), y(u, v, w), z(u, v, w))|J(u, v, w)|dudvdw,$$

where D' is the region in the u, v, w coordinate system that corresponds to the region D in the x, y, z coordinate system, and $J(u, v, w)$ is the Jacobian determinant of the transformation.

Changing of variables in triple integrals



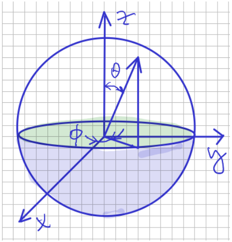
The Jacobian is given by:

$$J(u, v, w) \equiv \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

The Jacobian measures the change in volume due to the change of variables.

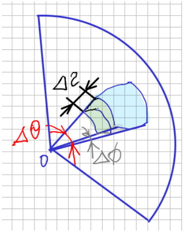
$$dV = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw.$$

Example of changing to the spherical coordinates



Let a function $f(x, y, z)$ be defined in Cartesian coordinates (x, y, z). Change to spherical coordinates (r, θ, ϕ), where r is the radial distance from the origin, θ is the polar angle measured from the positive z-axis, and ϕ is the azimuthal angle measured from the positive x-axis in the xy-plane.

Example of changing to the spherical coordinates



The transformation
from Cartesian coordinates
to spherical coordinates is given by:

$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$

where
$0 \le r \le R$, $0 \le \theta \le \pi$, and $0 \le \phi \le 2\pi$.

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta.$$

From the Fourier sum to the Fourier integral

Let $f(x)$ is defined as $x \in (-\infty, \infty)$ and $\int_{-\infty}^{\infty} |f(x)| dx = F$. Consider a Fourier series on the interval $(-T, T)$ be a periodic function with period T, and let a_n and b_n be its Fourier coefficients, defined by:

$$c_n = \frac{1}{2T} \int_{-T}^{T} f(y) \exp\left(-i \frac{\pi n}{T} y\right) dy$$

The Fourier series of $f(x)$ is then given by:

$$f(x) = \frac{c_0}{2} + \sum_{n=-\infty}^{\infty} \left(c_n \exp\left(i \frac{\pi n x}{T} \right) \right).$$

From the Fourier series to the Fourier integral

Using these expressions, we can write the Fourier series as:

$$f(x) = \frac{1}{2T}\int_{-T}^{T} f(y)dy +$$
$$\frac{1}{2T}\left(\sum_{n=-\infty}^{\infty}\int_{-T}^{T} f(y)\exp\left(-i\frac{\pi n}{T}y\right)dy\exp\left(i\frac{\pi n}{T}x\right)\right) =$$
$$\frac{1}{2T}\int_{-T}^{T} f(y)dy + \frac{1}{2T}\left(\sum_{n=-\infty}^{\infty}\int_{-T}^{T} f(y)\exp\left(i\frac{\pi n}{T}(x-y)\right)dy\right).$$

A sketch of derivation of the Fourier transform

Define $\Delta k = \frac{\pi}{T}$, $k_n = \frac{\pi}{T} n$.

$$f(x) = \frac{\Delta x}{2\pi} \int_{-T}^{T} f(y) dy + \sum_{n=-\infty}^{\infty} \left(\int_{-T}^{T} f(y) \exp\left(i k_n (x-y)\right) dy \right) \Delta k.$$

Consider a limit $T \to \infty \Rightarrow \Delta x \to 0$, then one gets:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y) \exp\left(ik(x-y)\right) dy \right) dk.$$

Let's rewrite in more convenient form:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y) \exp(-iky) dy \right) \exp(ikx) dk.$$

A sketch of derivation of the Fourier transform

Define

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-iky} dy.$$

If $f(y) \in C^2(-\infty, \infty)$ and $\int_{-\infty}^{\infty} \left| \frac{d^\alpha f}{dy^\alpha} \right| dy < \infty$, where $\alpha \in \{0, 1, 2\}$, then $\tilde{f}(k) \sim O(k^{-2})$.

Proof.

$$\int_{-\infty}^{\infty} f(y) e^{-iky} dy = \left. \frac{f(y) e^{-iky}}{ik} \right|_{-\infty}^{\infty} + \frac{1}{ik} \int_{-\infty}^{\infty} f'(y) e^{-iky} dy =$$

$$\left. \frac{f'(y) e^{-iky}}{-k^2} \right|_{-\infty}^{\infty} + \frac{1}{k^2} \int_{-\infty}^{\infty} f''(y) e^{-iky} dy = O(k^{-2}).$$

The Fourier transform

Therefore:

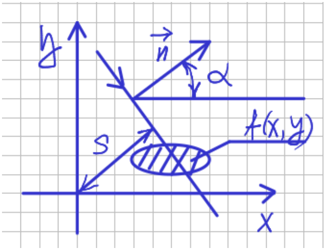
$$\int_{-\infty}^{\infty} \left| \hat{f}(k) e^{-ikx} \right| dx \leq \infty.$$

Denote:

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-iky} dy,$$
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk,$$

These integrals define the Fourier transform for the function $f(x)$ and inverse Fourier transform respectively.

The Radon transform



Let's consider a straight line with a normal $\vec{n} = (\cos(\alpha), \sin(\alpha))$ on the distance s from the origin: $x\cos(\alpha) + y\sin(\alpha) - s = 0$. The parametric form of this line is follow:

$$x = -\sin(\alpha)t + s\cos(\alpha),$$
$$y = \cos(\alpha)t + s\sin(\alpha).$$

Then integral alonn this straight line for the function $f(x, y)$ is follow:

$$R(s, \alpha) = \int_{-\infty}^{\infty} f(-t\sin(\alpha) + s\cos(\alpha), t\cos(\alpha) + s\sin(\alpha))dt.$$

The Radon transform and Fourier transform

The Radon transform can be represented trough two dimensional Fourier integral:

$$\hat{f}(\lambda_1, \lambda_2) = \int_{\mathbb{R}^2} f(x, y) e^{i(\lambda_1 x + \lambda_2 y)} dx\, dy,$$

$$\lambda_1 = \omega \cos(\alpha),\ \ \lambda_2 = \omega \sin(\alpha),$$

$$s = x\cos(\alpha) + y\sin(\alpha),\ \ t = -x\sin(\alpha) + y\cos(\alpha).$$

$$x = s\cos(\alpha) - t\sin(\alpha),\ \ y = s\sin(\alpha) + t\cos(\alpha)$$

$$\hat{f}(\omega\cos(\theta), \omega\sin(\theta)) =$$

$$\iint f(s\cos(\alpha) - t\sin(\alpha), s\sin(\alpha) + t\cos(\alpha)) e^{i\omega s} dt\, ds =$$

$$\int_{-\infty}^{\infty} e^{i\omega s} \int_{-\infty}^{\infty} f(s\cos(\alpha) - t\sin(\alpha), s\sin(\alpha) + t\cos(\alpha)) dt\, ds.$$

The inverse Radon transform

$$R(s,\alpha) = \int_{-\infty}^{\infty} \hat{f}(\omega\cos(\alpha), \omega\sin(\alpha))e^{-i\omega s}d\omega.$$

The inverse Fourier transform is given by formula:

$$f(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{f}(\lambda_1,\lambda_2)e^{i(\lambda_1 x+\lambda_2 y)}d\lambda_1 d\lambda_2 =$$

$$\frac{1}{4\pi^2}\int_{0}^{2\pi}\int_{0}^{\infty}\hat{f}(\omega\cos(\alpha),\omega\sin(\alpha))e^{i\omega(x\cos(\alpha)+y\sin(\alpha))}\omega d\omega d\alpha$$

$$\frac{1}{4\pi^2}\int_{0}^{2\pi}\int_{0}^{\infty}\hat{R}(\omega.\alpha)e^{i\omega(x\cos(\alpha)+y\sin(\alpha))}\omega d\omega d\alpha$$

where $\hat{R}(\omega.\alpha) = \int_{-\infty}^{\infty} R(s,\alpha)e^{i\omega s}ds$.

The Radon transform and inverse Radon transform

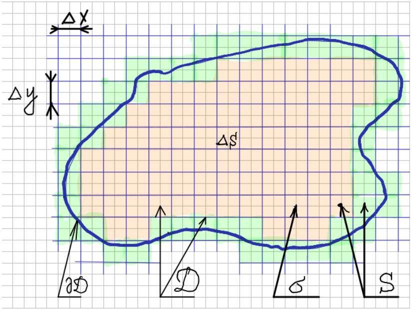
$$\hat{f}(\lambda_1, \lambda_2) = \int_{\mathbb{R}^2} f(x, y) e^{i(\lambda_1 x + \lambda_2 y)} dx\, dy,$$

$$R(s, \alpha) = \int_{-\infty}^{\infty} \hat{f}(\omega \cos(\alpha), \omega \sin(\alpha)) e^{-i\omega s} d\omega.$$

$$f(x, y) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{\infty} \hat{R}(\omega.\alpha) e^{i\omega(x\cos(\alpha) + y\sin(\alpha))} \omega d\omega d\alpha,$$

$$\hat{R}(\omega.\alpha) = \int_{-\infty}^{\infty} R(s, \alpha) e^{i\omega s} ds.$$

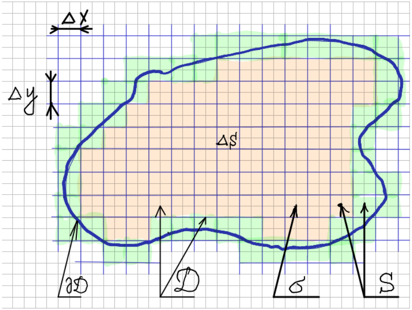
2D mesh



Let $f(x, y)$ be a continuous function defined on a domain $\mathcal{D}$. The border of the domain $\partial\mathcal{D}$ is a piecewise smooth curve. One common method is to use a rectangular grid with N_x points in the x-direction and N_y points in the y-direction.

Let's define $\max_{x\in\mathcal{D}} = b,\ \min_{x\in\mathcal{D}} = a,\ \max_{y\in\mathcal{D}} = \beta,\ \min_{y\in\mathcal{D}} = \alpha$; $\Delta x = (b-a)/N_x,\ \Delta y = (\beta-\alpha)/N_y$
Define the internal rectangles S_k and the rectangles intersected the border of the domain as s_k

An approximation of 2D integral



Consider the approximation of the integral by the sum:

$$\iint_{\mathcal{D}} f(x,y)\,dS \approx \sum_{j=1}^{N} f_k \Delta S + \frac{1}{2}\sum_{l=1}^{n} f_l \Delta S,$$

where $\Delta S = \Delta x \Delta y$ and $f_k = f(x_i, y_j)$. The point(x_i, y_j) is the center of the rectangle S_k and $f_l = f(x_i, y_j)$, where (x_i, y_j) is some point of s_l.

Error estimation

To estimate the error in the numerical approximation of the integral using the rectangular grid method, we use Taylor's theorem to write the function $f(x, y)$ as:

$$f(x,y) = f(x_i,y_j) + \frac{\partial f}{\partial x}(x_i,y_j)(x-x_i) + \frac{\partial f}{\partial y}(x_i,y_j)(y-y_j) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x_i,y_j)(x-x_i)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(x_i,y_j)(y-y_j)^2 + \frac{\partial^2 f}{\partial x \partial y}(x_i,y_j)(x-x_i)(y-y_j) + O(\Delta x^3, \Delta y^3),$$

where x_i and y_j are the coordinates of the grid point (i, j), and $O(\Delta x^3, \Delta y^3)$ denotes terms of order Δx^3 and Δy^3 or higher.

The error estimation in an elementary rectangle

Integrating both sides of this equation over the rectangle S_i and using the fact that the integral of any odd function over a symmetric interval is zero, we obtain:

$$\left| \iint_{\mathcal{S}_i} f(x,y)\,dS - f(x_i,y_j)\Delta S \right| \leq$$

$$\left| \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(x_i,y_j)\frac{2}{3}\left(\frac{\Delta x}{2}\right)^3 + \frac{\partial^2 f}{\partial y^2}(x_i,y_j)\frac{2}{3}\left(\frac{\Delta y}{2}\right)^3\right) + \mathcal{O}((\Delta x)^4,(\Delta y)^4)\right) \right|.$$

The error estimation

$$\left| \iint_{\mathcal{S}_i} f(x,y)\, dS - \sum_{k=1}^{N} f_k \Delta S - \frac{1}{2} \sum_{l=1}^{n} f_l \Delta S \right| \leq \mathcal{O}\left(\frac{MS}{N^2}\right).$$

Here M is maximal absolute value of second derivatives of $f(x,y)$ on the domain $\mathcal{D}$. S is the area of $\mathcal{D}$. N – number of the mesh rectangles in $\mathcal{D}$.

The error estimation. An example.

$$F = \int_0^{\pi/2} \int_0^1 (1 - r^2) r \, dr \, d\phi \sim 0.392699$$

$$\begin{aligned}
N = 10, &\quad S \sim 0.3687, \ (F - S)N^2 = 2.4; \\
N = 20, &\quad S \sim 0.3863, \ (F - S)N^2 = 2.56; \\
N = 50, &\quad S \sim 0.39169, \ (F - S)N^2 = 2.51; \\
N = 100, &\quad S \sim 0.3924, \ (F - S)N^2 = 2.74; \\
N = 200, &\quad S \sim 0.3926, \ (F - S)N^2 = 2.68.
\end{aligned}$$

Discrete Fourier transform

Consider vector f_n of values of the function $f(x)$ in the points $x_n = x_0 + k\frac{x_N - x_0}{N}n,\ n \in \{\overline{0, N-1}\}$
The discrete Fourier transform (DFT) of a sequence of length N is defined as:

$$\tilde{f}_k = \sum_{n=0}^{N-1} f_n e^{-2\pi i \frac{nk}{N}}$$

$\tilde{f}_k$ is the k th sample of the DFT output sequence, and i is the imaginary unit.

Inverse discrete Fourier transform

The inverse discrete Fourier transform (IDFT) is the mathematical operation that takes a sequence of equally-spaced samples of the discrete-time Fourier transform (DTFT) and transforms it back into a sequence of samples of a function.
The IDFT of a sequence of length N is defined as:

$$f_n = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{f}_k e^{2\pi i \frac{nk}{N}}$$

where $\tilde{f}_k$ is the kth sample of the input sequence in the frequency domain, and f_n is the nth sample of the output sequence in the time domain.