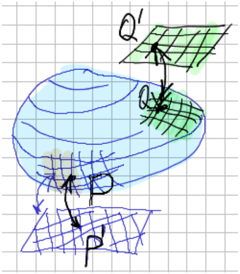
Surface integrals and Ostrogradsky-Gauss theorem

O.M. Kiselev
o.kiselev@innopolis.ru

Innopolis university

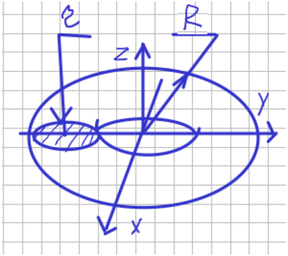
April 28, 2023

Embedded 2D manifolds in 3-dimensional space



An embedded manifold is a subset M of Euclidean space $\mathbb{R}^3$ that can be locally parameterized by a smooth function $\mathbf{x} : U \to \mathbb{R}^3$, where U is an open subset of $\mathbb{R}^2$. In particular, for any point $\mathbf{p} \in M$, there exists a neighborhood V of $\mathbf{p}$ in $\mathbb{R}^3$ and a smooth function $\mathbf{x} : U \to V \cap M$ such that $\mathbf{x}$ is a homeomorphism between U and $\mathbf{x}(U) \subseteq M$.

An example of embedded 2D manifold

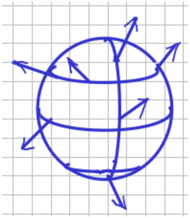


The parametric equation for a torus with major radius R and minor radius r is given by:

$$x = (R + r\cos\theta)\cos\phi$$
$$y = (R + r\cos\theta)\sin\phi$$
$$z = r\sin\theta$$

where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq 2\pi$ are the polar and azimuthal angles, respectively.

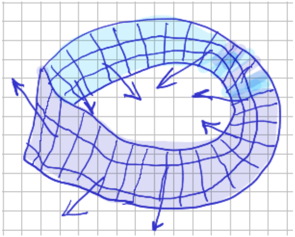
Orientable manifolds



An orientable manifold is a regular manifold M that admits a consistent choice of orientation. More precisely, M is orientable if and only if there exists a continuous non-vanishing vector field $\mathbf{v}$ on M such that for any two points $\mathbf{p}, \mathbf{q} \in M$, the parallel transport of $\mathbf{v}(\mathbf{p})$ along any smooth path connecting $\mathbf{p}$ and $\mathbf{q}$ is equal to $\mathbf{v}(\mathbf{q})$.

As a typical example one can image a sphere.

Non-orientable manifolds



A non-orientable manifold is a regular manifold M that does not admit a consistent choice of orientation. More precisely, M is non-orientable if and only if there exists a continuous non-vanishing vector field $\mathbf{v}$ on M such that for any two points $\mathbf{p}, \mathbf{q} \in M$, the parallel transport of $\mathbf{v}(\mathbf{p})$ along any closed loop on M is equal to $-\mathbf{v}(\mathbf{p})$. A classic example of a non-orientable manifold is the Möbius strip.

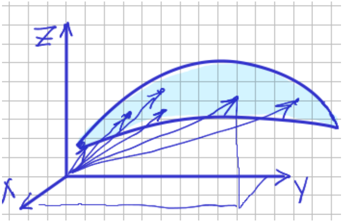
A parametric definition of the M"obius strip

The Möbius strip can be parametrized by the following equations:

$$x(u,v) = \left(1 + \frac{v}{2}\cos\frac{u}{2}\right)\cos u$$
$$y(u,v) = \left(1 + \frac{v}{2}\cos\frac{u}{2}\right)\sin u$$
$$z(u,v) = \frac{v}{2}\sin\frac{u}{2}$$

where $0 \le u \le 2\pi$ and $-1 \le v \le 1$.
In these equations, the parameter u controls the orientation of the strip around its central axis, while the parameter v controls the width of the strip. The strip has a half-twist in it, which can be seen by observing that the z-coordinate changes sign as u goes from 0 to 2π.

A definitions of a surface using a vector approach

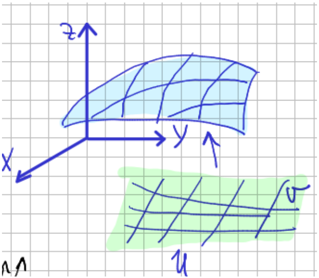


A parametric surface
is a surface in three-dimensional
space that is defined using
a set of equations of the form:

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

where u and v are parameters that vary over some domain,
and $\vec{i}$, $\vec{j}$, and $\vec{k}$ are the standard basis vectors in
three-dimensional space.

A parametric definition of a surface using a coordinate approach



Alternatively, a parametric surface can be defined using a set of three equations of the form:

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)$$

where u and v vary over some domain. These equations describe how the x, y, and z coordinates of a point on the surface vary as the parameters u and v vary.

A tangent plane for given surface

Let S be a surface in $\mathbb{R}^3$ given by the equation $z = f(x, y)$, where f is a differentiable function.
The equation of the tangent plane is given by:

$$z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$

or, equivalently,

$$\mathbf{r}(x, y) = \langle x, y, f(a, b)\rangle +$$

$$\langle \frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b), -1\rangle \cdot \langle x - a, y - b, f(x, y) - f(a, b)\rangle = 0$$

An area of surfaces

The area of a small piece of the surface can be approximated using the formula for the area of a parallelogram:

$$dA = ||\vec{r_u} \times \vec{r_v}||dudv$$

where $\vec{r_u}$ and $\vec{r_v}$ are tangent vectors to the surface, and du and dv are small increments in the u and v directions, respectively. To get the total area of the surface, we integrate this formula over the entire surface:

$$A = \iint_S ||\vec{r_u} \times \vec{r_v}||dudv$$

where S is the surface we're interested in and the double integral is taken over a parametrization of the surface.

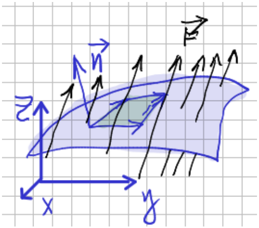
An integral over a surface

Integration on a surface is the process of calculating the integral of a function $f(x, y, z)$ over a two-dimensional surface S in three-dimensional space. The surface S can be defined using either a set of equations or a parametrization, and the integral is typically computed using a double integral over a parametrization of the surface:

$$\int \int_S f(x, y, z)\, dS$$

where dS represents the area element on the surface S.

An integral over a surface for a vector field



Let S be a smooth oriented surface in $\mathbb{R}^3$ with unit normal vector field $\hat{n}$. Let

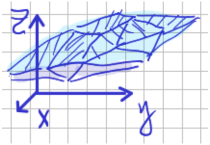
$$\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$$

be a continuous vector field defined on a region containing S. Then the flux of $\mathbf{F}$ across S is given by:

$$\iint_S \mathbf{F} \cdot \vec{n}\, dS$$

where $\vec{n}$ is the unit normal vector to S, and dS is the differential surface area element on S.

A geometric interpretation of surface integral



Let $f(\mathbf{r})$ be a scalar field and $\mathbf{F}(\mathbf{r})$ be a vector field defined in the region of 3D space containing the surface S. Let S_Δ be a collection of small patches or elements of the surface S, such that the union of all the patches covers the entire surface. Each patch S_i has an area ΔS_i, and is centered at a point $\mathbf{r}_i$ on the surface.

The surface integral of f over S can be approximated by a sum of integrals over each small patch, weighted by the area of the patch:

$$\iint_S f(\mathbf{r})\, dS = \lim_{\Delta S_i \to 0} \sum_i f(\mathbf{r}_i) \Delta S_i$$

A geometric interpretation of surface integral

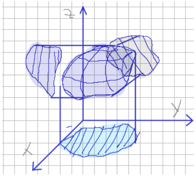
Similarly, the surface integral of the vector field $\mathbf{F}$ over S can be approximated by a sum of integrals over each small patch, weighted by the normal vector to the patch:

$$\iint_S \mathbf{F}(\mathbf{r}) \cdot d\mathbf{S} \approx \sum_i \mathbf{F}(\mathbf{r}_i) \cdot \mathbf{n}_i \Delta S_i$$

where $\mathbf{n}_i$ is the outward-pointing normal vector to the patch S_i.

$$\iint_S \mathbf{F}(\mathbf{r}) \cdot d\mathbf{S} = \lim_{\Delta S_i \to 0} \sum_i \mathbf{F}(\mathbf{r}_i) \cdot \mathbf{n}_i \Delta S_i$$

A surface integral as an integral over projections



$$\iint_S \mathbf{F}\cdot\vec{n}\,dS = \iint_S (F_1(x,y,z)n_1 + F_2(x,y,z)n_2 + F_3(x,y,z)n_3)\,dS.$$

The projection of the infinitesimal area dS on coordinate planes can be represented as follows: $n_1 dS = dy\,dz$, $n_2 dS = dz\,dx$, $n_3 dS = dx\,dy$.
As a result one obtains:

$$\iint_S \mathbf{F}\cdot\vec{n}\,dS = \iint_S F_1(x,y,z)dy\,dz + F_2(x,y,z)dz\,dx + F_3(x,y,z)dx\,dy.$$

The formula written in the terms of the cross product

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D (\mathbf{F}(\mathbf{r}(u,v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u}(u,v)) \times \frac{\partial \mathbf{r}}{\partial v}(u,v) \right) \, du \, dv$$

In this formula, $\mathbf{F}$ is the vector field, $\mathbf{r}(u,v)$ is the parameterization of the surface S, and D is the domain in the (u,v)-plane over which the surface is parameterized.

$$d\mathbf{S} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \, du \, dv.$$

An example of the surface integral

Compute the surface integral of the vector field $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ over the part of the surface of the sphere $x^2 + y^2 + z^2 = 4$ that lies in the first octant.
One possible parameterization of the surface of the sphere is given by:

$$\mathbf{r}(\theta, \phi) = (2\sin\theta\cos\phi)\mathbf{i} + (2\sin\theta\sin\phi)\mathbf{j} + (2\cos\theta)\mathbf{k}$$

where $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \phi \leq \frac{\pi}{2}$ are the polar and azimuthal angles, respectively, that specify the location of each point on the surface in the first octant.

An example of the surface integral

To evaluate the surface integral, we need to compute the dot product of the vector field $\mathbf{F}$ with the area element $d\mathbf{S}$ at each point on the surface, and then integrate over the surface using the appropriate area element:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \mathbf{F}(\mathbf{r}(\theta, \phi)) \cdot (\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi})\, d\theta\, d\phi$$

where $\frac{\partial \mathbf{r}}{\partial \theta}$ and $\frac{\partial \mathbf{r}}{\partial \phi}$ are the partial derivatives of $\mathbf{r}(\theta, \phi)$ with respect to θ and ϕ, respectively, and $\times$ denotes the cross product.

An example of the surface integral

Using the parameterization $\mathbf{r}(\theta, \phi)$ and the definition of the vector field $\mathbf{F}$, we can compute:

$$\frac{\partial \mathbf{r}}{\partial \theta} = 2\cos(\theta)\cos(\phi)\mathbf{i} + 2\cos(\theta)\sin(\phi)\mathbf{j} - 2\sin(\theta)\mathbf{k},$$

$$\frac{\partial \mathbf{r}}{\partial \phi} = -2\sin(\theta)\sin(\phi)\mathbf{i} + 2\sin(\theta)\cos(\phi)\mathbf{j} + 0\mathbf{k},$$

$$\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos(\theta)\cos(\phi) & 2\cos(\theta)\sin(\phi) & -2\sin(\theta)\mathbf{k} \\ -2\sin(\theta)\sin(\phi) & 2\sin(\theta)\cos(\phi) & 0 \end{vmatrix} =$$

$$4\sin^2\theta\cos\phi\mathbf{i} + 4\sin^2\theta\sin\phi\mathbf{j} + 2\sin(2\theta)\mathbf{k},$$

$$\mathbf{F}(\mathbf{r}(\theta, \phi)) = 4\sin^2\theta\cos^2\phi\mathbf{i} + 4\sin^2\theta\sin^2\phi\mathbf{j} + 4\cos^2\theta\mathbf{k}.$$

An example of the surface integral

Substituting these expressions into the surface integral, we get:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (16 \sin(\theta)^4 \sin(\phi)^3 +$$

$$4 \cos(\theta)^2 \left(4 \cos(\theta) \sin(\theta) \sin(\phi)^2 + 4 \cos(\theta) \sin(\theta) \cos(\phi)^2\right) +$$

$$16 \sin(\theta)^4 \cos(\phi)^3)\, d\theta\, d\phi =$$

$$\int_0^{\pi/2} (3\pi \sin(\phi)^3 + 4 \sin(\phi)^2 + 3\pi \cos(\phi)^3 + 4 \cos(\phi)^2) d\phi = 6\pi$$

The Ostrogradsky-Gauss therem

For a smooth vector field $\mathbf{F}(\mathbf{r})$ defined in a region of 3D space containing a closed surface S that encloses a volume V, we have:

$$\iiint_V (\nabla \cdot \mathbf{F})\, dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$

In this formula, V is a closed volume, with boundary surface S, and $\mathbf{F}$ is a vector field.

$$\iiint_V (\nabla \cdot \mathbf{F})\, dV = \iiint_V \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} dx\, dy\, dz$$

A proof of the Ostrogradsky-Gauss theorem

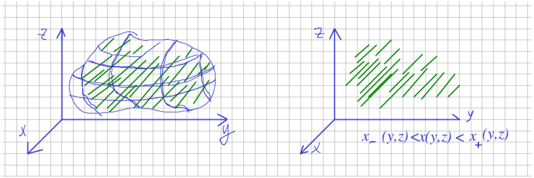


Figure: The integration over alone $x_-(y,z) < x < x_+(y,z)$ looks like changing the body onto **a bunch of spaghetti**. Here the bunch of spaghetti are noted as a set of green intervals.

Let's consider the first term:

$$\iiint_V \frac{\partial F_1}{\partial x} dx\, dy\, dz = \iint_{\partial V} \int_{x_-(y,z)}^{x_+(y,z)} \frac{\partial F_1}{\partial x} dx\, dy\, dz =$$

$$\iint_{\partial V^+} F_1(x_+, y, z) dy\, dz - \iint_{\partial V^-} F_1(x_-, y, z) dy\, dz = \iint_{\partial V} F_1(x, y, z) dy\, dz.$$

A proof of the Ostrogradsky-Gauss theorem

Similar calculations for the rest parts of the integral give a result:

$$\iiint_V (\nabla \cdot \mathbf{F})\, dV$$

$$= \iint_{\partial V} F_1(x, y, z) dy\, dz + F_2(x, y, z) dz\, dx + F_3(x, y, z) dx\, dy = \iint_{\partial V} \mathbf{F} \cdot d\mathbf{S}.$$

Ostrogradsky-Gauss theorem. An example

Compute integral $\iint_{\partial C} x^2 dy\,dz + y^2 dz\,dx + z^2 dy\,dx$, where ∂C is a surface of cone $\frac{x^2}{a^2} + \frac{y^2}{a^2} = \frac{z^2}{b^2}$ as $0 < z < b$.
Let's change the variables $x = \frac{a}{b} r\cos(\alpha), y = \frac{a}{b} r\sin(\alpha), z = z$.
Then one gets:

$$\iint_{\partial C} x^2 dy\,dz + y^2 dz\,dx + z^2 dy\,dx =$$

$$\iiint_C (2x + 2y + 2z) dx\,dy\,dz =$$

$$\int_0^b \int_0^{2\pi} \int_0^z (2r\cos(\phi) + 2r\sin(\phi) + 2z)\frac{a^2}{b^2} r\,dr\,d\phi\,dz =$$

$$4\pi\frac{a^2}{b^2}\int_0^b \int_0^z rzdr\,dz = \frac{\pi}{2} a^2 b^2.$$

A physical interpretation of the Ostrogradsky-Gauss theorem

Consider a fluid flowing through a closed surface $\mathcal{S}$ in three-dimensional space. The velocity of the fluid at a point (x, y, z) is given by the vector field $\mathbf{v}(x, y, z) = v_x(x, y, z)\mathbf{i} + v_y(x, y, z)\mathbf{j} + v_z(x, y, z)\mathbf{k}$.
The divergence of this vector field represents the rate at which fluid is flowing out of a given volume:

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

A physical interpretation of the Ostrogradsky-Gauss theorem

The divergence theorem states that the total amount of fluid flowing out of the closed surface $\mathcal{S}$ is equal to the integral of the divergence of the velocity field over the volume enclosed by the surface:

$$\iint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{S} = \iiint_{V} (\nabla \cdot \mathbf{v})\, dV$$

where $d\mathbf{S}$ is the outward-pointing differential surface element on $\mathcal{S}$ and dV is the differential volume element inside the surface.

rot of a vector field as a limit of circulation

The rotor (also known as the curl) of a vector field $\mathbf{F}$ at a point is a measure of how much the vector field "curls" or rotates around that point. One way to express the rotor using the Green's formula is as follows:

$$\mathrm{rot}(\mathbf{F})(\mathbf{r}) = \lim_{A \to 0} \frac{1}{A} \oint_C \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}'$$

where $\mathbf{r}$ is the point of interest, C is a small closed curve centered at $\mathbf{r}$, and A is the area enclosed by C.

rot of a vector field as a limit of circulation

This formula tells us that the rotor of **F** at a point **r** is equal to the limit of the circulation of **F** around a small closed curve C centered at **r**, as the area enclosed by C shrinks to zero. To see why this formula is true, we can use the Green's formula, which relates the circulation of a vector field around a closed curve to the integral of the rotor of the vector field over the area enclosed by the curve:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_A \mathrm{rot}(\mathbf{F}) \cdot d\mathbf{S}$$

where A is the area enclosed by the closed curve C, and $d\mathbf{S}$ is the outward-pointing differential surface element on A.

rot of a vector field as a limit of circulation

If we divide both sides of this equation by the area A and take the limit as $A \to 0$, we obtain:

$$\lim_{A\to 0} \frac{1}{A} \oint_C \mathbf{F} \cdot d\mathbf{r} = \lim_{A\to 0} \frac{1}{A} \iint_A \mathrm{rot}(\mathbf{F}) \cdot d\mathbf{S}$$

The left-hand side of this equation is the circulation of **F** around a small closed curve C centered at **r**, and the right-hand side is the average value of the rotor of **F** over the area enclosed by C.

rot of a vector field as a limit of circulation

Therefore, as A shrinks to zero, the right-hand side approaches the value of the rotor of $\mathbf{F}$ at $\mathbf{r}$, and we obtain the formula:

$$\mathrm{rot}(\mathbf{F})(\mathbf{r}) = \lim_{A \to 0} \frac{1}{A} \oint_C \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}'$$

This formula tells us that the rotor of $\mathbf{F}$ at a point $\mathbf{r}$ is equal to the limit of the circulation of $\mathbf{F}$ around a small closed curve C centered at $\mathbf{r}$, as the area enclosed by C shrinks to zero.

A divergence as a limit of a flow through an envelope

Let's consider a three-dimensional vector field $\mathbf{F}(\mathbf{r}) = F_x(\mathbf{r})\mathbf{i} + F_y(\mathbf{r})\mathbf{j} + F_z(\mathbf{r})\mathbf{k}$ and a small closed envelope $\mathcal{E}$ centered at a point $\mathbf{r}_0$ in space.

We can think of the envelope $\mathcal{E}$ as a small smooth surface enveloping a volume ϵ.

The net flow rate of $\mathbf{F}$ through the closed envelope $\mathcal{E}$ is given by the flux integral:

$$\text{Flow} = \iint_{\partial\mathcal{E}} \mathbf{F} \cdot d\mathbf{S}$$

where $\partial\mathcal{E}$ is the boundary surface of the envelope $\mathcal{E}$ and $d\mathbf{S}$ is the outward-pointing differential surface element on $\partial\mathcal{E}$.

A divergence as a limit of a flow through an envelope

By the divergence theorem, the net flow rate of $\mathbf{F}$ through the closed envelope $\mathcal{E}$ is equal to the integral of the divergence of $\mathbf{F}$ over the volume enclosed by the envelope:

$$\iint_{\partial\mathcal{E}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{E}} \nabla \cdot \mathbf{F} \, dV$$

where dV is the differential volume element inside the envelope $\mathcal{E}$.

A divergence as a limit of a flow through an envelope

As the size of the envelope $\mathcal{E}$ shrinks to zero, we can define the divergence of $\mathbf{F}$ at $\mathbf{r}_0$ as the limit of the net flow rate through small closed envelopes centered at $\mathbf{r}_0$, as the size of the envelopes shrinks to zero:

$$\mathrm{div}(\mathbf{F})(\mathbf{r}_0) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \iiint_{\mathcal{E}} \nabla \cdot \mathbf{F}\, dV = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \iint_{\partial \mathcal{E}} \mathbf{F} \cdot d\mathbf{S}$$

where $\mathcal{E}$ is a small closed envelope centered at $\mathbf{r}_0$ and enclosing a volume ϵ.