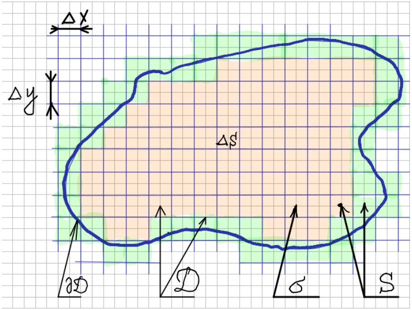
Properties of curves in three dimensional space

O.M. Kiselev
o.kiselev@innopolis.ru

Innopolis university

April 14, 2023

2D mesh

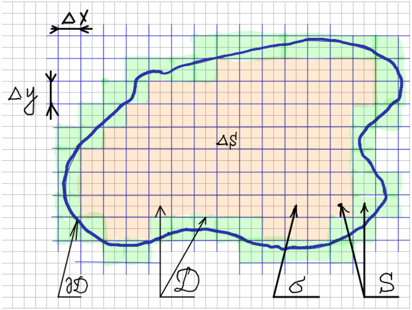


Let $f(x, y)$ be a continuous function defined on a domain $\mathcal{D}$. The border of the domain $\partial\mathcal{D}$ is a piecewise smooth curve. One common method is to use a rectangular grid with N_x points in the x-direction and N_y points in the y-direction.

Let's define $\max_{x\in\mathcal{D}} = b,\ \min_{x\in\mathcal{D}} = a,\ \max_{y\in\mathcal{D}} = \beta,\ \min_{y\in\mathcal{D}} = \alpha$; $\Delta x = (b-a)/N_x,\ \Delta y = (\beta-\alpha)/N_y$

Define the internal rectangles S_k and the rectangles intersected the border of the domain as s_k

An approximation of 2D integral



Consider the approximation of the integral by the sum:

$$\iint_{\mathcal{D}} f(x,y)\,dS \approx$$

$$\sum_{j=1}^{N} f_k \Delta S + \frac{1}{2}\sum_{l=1}^{n} f_l \Delta S,$$

where $\Delta S = \Delta x \Delta y$ and $f_k = f(x_i, y_j)$. The point(x_i, y_j) is the center of the rectangle S_k and $f_l = f(x_i, y_j)$, where (x_i, y_j) is some point of s_l.

Error estimation

To estimate the error in the numerical approximation of the integral using the rectangular grid method, we use Taylor's theorem to write the function $f(x, y)$ as:

$$\begin{aligned} f(x, y) = f(x_i, y_j) + \frac{\partial f}{\partial x}(x_i, y_j)(x - x_i) + \frac{\partial f}{\partial y}(x_i, y_j)(y - y_j) + \\ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x_i, y_j)(x - x_i)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(x_i, y_j)(y - y_j)^2 + \\ \frac{\partial^2 f}{\partial x \partial y}(x_i, y_j)(x - x_i)(y - y_j) + O(\Delta x^3, \Delta y^3), \end{aligned}$$

where x_i and y_j are the coordinates of the grid point (i, j), and $O(\Delta x^3, \Delta y^3)$ denotes terms of order Δx^3 and Δy^3 or higher.

The error estimation in an elementary rectangle

Integrating both sides of this equation over the rectangle S_i and using the fact that the integral of any odd function over a symmetric interval is zero, we obtain:

$$\left| \iint_{\mathcal{S}_i} f(x, y)\, dS - f(x_i, y_j)\Delta S \right| \leq$$
$$\left| \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(x_i, y_j)\frac{2}{3}\left(\frac{\Delta x}{2}\right)^3 + \frac{\partial^2 f}{\partial y^2}(x_i, y_j)\frac{2}{3}\left(\frac{\Delta y}{2}\right)^3\right) + \right.$$
$$\left. \mathcal{O}((\Delta x)^4, (\Delta y)^4) \Big) \right|.$$

The error estimation

$$\left| \iint_{\mathcal{S}_i} f(x,y)\, dS - \sum_{k=1}^{N} f_k \Delta S - \frac{1}{2} \sum_{l=1}^{n} f_l \Delta S \right| \leq \mathcal{O}\left(\frac{MS}{N^2}\right).$$

Here M is maximal absolute value of second derivatives of $f(x,y)$ on the domain $\mathcal{D}$. S is the area of $\mathcal{D}$. N – number of the mesh rectangles in $\mathcal{D}$.

The error estimation. An example.

$$F = \int_0^{\pi/2} \int_0^1 (1 - r^2) r \, dr \, d\phi \sim 0.392699$$

$$\begin{aligned}
N = 10, &\quad S \sim 0.3687,\ (F - S)N^2 = 2.4;\\
N = 20, &\quad S \sim 0.3863,\ (F - S)N^2 = 2.56;\\
N = 50, &\quad S \sim 0.39169,\ (F - S)N^2 = 2.51;\\
N = 100, &\quad S \sim 0.3924,\ (F - S)N^2 = 2.74;\\
N = 200, &\quad S \sim 0.3926,\ (F - S)N^2 = 2.68.
\end{aligned}$$

Discrete Fourier transform

Consider vector f_n of values of the function $f(x)$ in the points $x_n = x_0 + k\frac{x_N - x_0}{N}n,\ n \in \{\overline{0, N-1}\}$
The discrete Fourier transform (DFT) of a sequence of length N is defined as:

$$\tilde{f}_k = \sum_{n=0}^{N-1} f_n e^{-2\pi i \frac{nk}{N}}$$

$\tilde{f}_k$ is the k th sample of the DFT output sequence, and i is the imaginary unit.

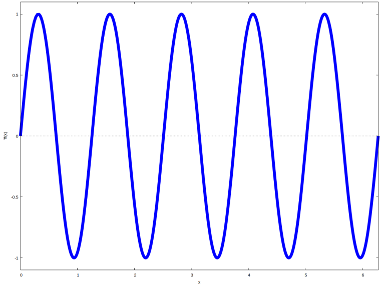
Inverse discrete Fourier transform

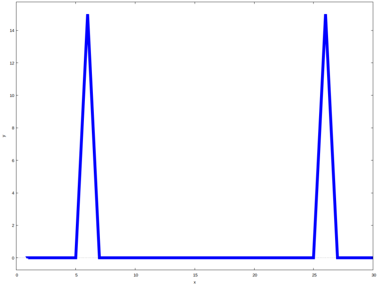
The inverse discrete Fourier transform (IDFT) is the mathematical operation that takes a sequence of equally-spaced samples of the discrete-time Fourier transform (DTFT) and transforms it back into a sequence of samples of a function.
The IDFT of a sequence of length N is defined as:

$$f_n = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{f}_k e^{2\pi i \frac{nk}{N}}$$

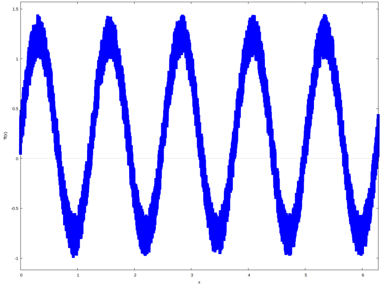
where $\tilde{f}_k$ is the kth sample of the input sequence in the frequency domain, and f_n is the nth sample of the output sequence in the time domain.

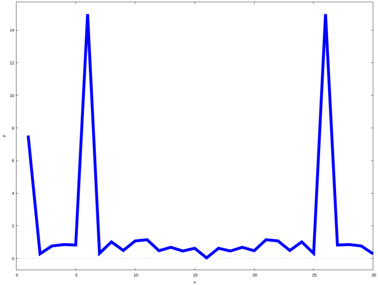
DFT. Example. $f(x) = \sin(5x)$



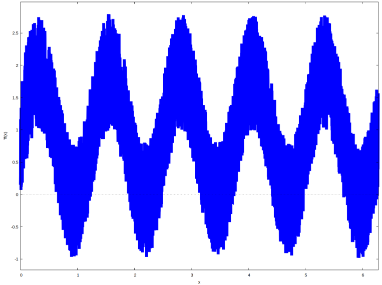


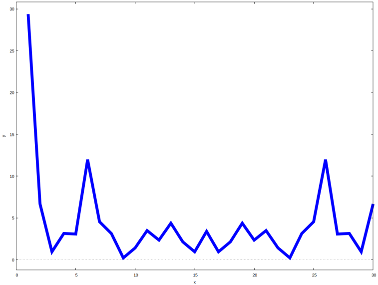
DFT. Example. $f(x) = \sin(5x) + 0.5 \cdot \text{rand}$





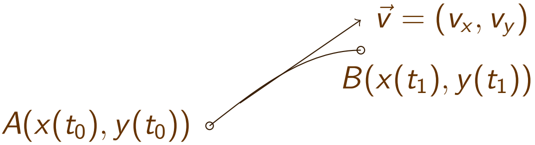
DFT. Example. $f(x) = \sin(5x) + 2 \cdot \text{rand}$





A parametric form of a curve

Let us consider a curve on a plane. Assume that in the Cartesian coordinates can be written as $x = x(t)$ and $y = y(t)$.



A length of a curve

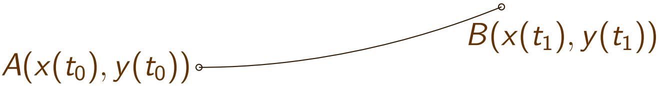
The components of tangent vector at given point can be defined as the derivatives with respect to t $v_x = \dot{x}$, $v_y = \dot{y}$. The length of the tangent vector:

$$V = \sqrt{v_x^2 + v_y^2}.$$

The length of the path for the curve of the point over the interval of the parameter $t \in [t_0, t_1]$:

$$S = \int_{t_0}^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2} dt.$$

Examples



Consider the plane curve:

$$x(t) = v_1 t, \quad y(t) = at^2.$$

The instant tangent vector at t is:

$$v_x = v_1, \quad v_y = 2at.$$

Examples

We can thought that the $x(t)$ and $y(t)$ is the parametric form for the curve of the point. In that way we obtain, that the vector of instant tangent vector at t. The instant tangent vector:

$$V(t) = \sqrt{v_1^2 + 4a^2t^2}.$$

The length of the length of the curve at t_1 is:

$$S(t_1) \quad = \int_0^{t_1} \sqrt{v_1^2 + 4a^2t^2} dt = \tfrac{t}{2}\sqrt{v_1^2 - 4a^2t^2} + \tfrac{v_1^2}{4a} \operatorname{asinh}\left(\tfrac{2at_1}{v_1}\right).$$

Integrating

$$S(t_1) = \int_0^{t_1} \sqrt{v_1^2 + 4a^2 t^2} dt = v_1 \int_0^{t_1} \sqrt{1 + \frac{4a^2}{v_1^2} t^2} dt =$$

$$\left| \frac{2a}{v_1} t = \sinh(\tau),\ \tau_1 = \operatorname{asinh}(2at_1/v_1),\ dt = \frac{v_1}{2a} \cosh(\tau) d\tau \right| =$$

$$\frac{v_1^2}{2a} \int_0^{\tau_1} \sqrt{1 + \sinh^2(\tau)} \cosh(\tau) d\tau = \frac{v_1^2}{2a} \int_0^{\tau_1} \cosh^2(\tau) d\tau =$$

Integrating

$$\frac{v_1^2}{8a}\int_0^{\tau_1}(e^{2\tau}+2+e^{-2\tau})d\tau = \frac{v_1^2}{4a}\tau_1 + \frac{v_1^2}{8a}\sinh(2\tau_1)$$

$$\frac{v_1^2}{4a}\tau_1 + \frac{v_1^2}{8a}2\sqrt{1-\sinh^2(\tau_1)}\,\sinh(\tau_1) =$$

$$\frac{v_1^2}{4a}\,\mathrm{asinh}\left(\frac{2at_1}{v_1}\right) + \frac{v_1^2}{4a}\sqrt{1-\frac{4a^2}{v_1^2}t^2}\frac{2a}{v_1}t$$

A curvature

The second derivative at the point:

$$a_x = \dot{v}_x = \ddot{x}, \quad a_y = \dot{v}_y = \ddot{y}.$$

Theorem

If $\sqrt{v_x^2 + v_y^2} = \mathrm{const}$, then the vector of the second derivative always is orthogonal to the tangent vector.

A curvature

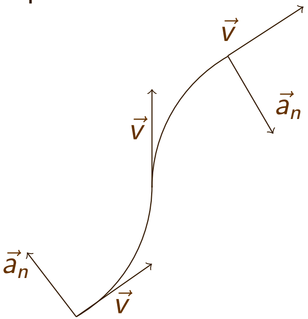
Proof. Let us differentiate the scalar product:

$$\frac{d}{dt}(\vec{v}, \vec{v}) = 0,$$

$$\left(\frac{d}{dt}\vec{v}, \vec{v}\right) + \left(\vec{v}, \frac{d}{dt}\vec{v}\right) = 0$$

$$2\left(\frac{d}{dt}\vec{v}, \vec{v}\right) = 0$$

$$(\vec{a}, \vec{v}) = 0.$$



A first derivative and tangent vector for the circle

Let us consider the circle:

$$x = R\cos(\omega t), \quad y = R\sin(\omega t).$$

The tangent vector is:

$$v_x = -R\omega\sin(\omega t), \quad v_y = R\omega\cos(\omega t).$$

The formula for the length of the tangent line looks like:

$$V = \sqrt{R^2\omega^2\sin^2(\omega t) + R^2\omega^2\cos^2(\omega t)} = R\omega.$$

Second derivative for the circle

The second derivative is defined the following formulas:

$$a_x = -R\omega^2 \cos(\omega t), \quad a_y = -R\omega^2 \sin(\omega t).$$

and

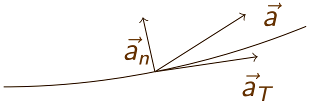
$$|a_n| = \sqrt{a_x^2 + a_y^2} = R\omega^2 = \frac{V^2}{R}.$$

This vector is orthogonal with respect to the tangent one. Therefore one obtains a *normal* vector.

$$\frac{1}{R} = \frac{|a_n|}{V^2}.$$

The quantity $\rho = 1/R$ is called a curvature.

Second derivative in general case



The second derivative might be represented as a sum two orthogonal vectors as the tangent direction and the normal one. The value of the tangent content of the second derivative can be obtained as follows:

$$|a_T| = \frac{(\vec{a}, \vec{v})}{\sqrt{(\vec{v}, \vec{v})}}.$$

The projection of the second vector of the tangent line can be represented as follows:

$$\vec{a}_T = \frac{(\vec{a}, \vec{v})}{(\vec{v}, \vec{v})}\vec{v} = \frac{a_x v_x + a_y v_y}{v_x^2 + v_y^2}(v_x \vec{i} + v_y \vec{j}).$$

Normal vector

The normal vector can be represented as:

$$\vec{a}_n = \vec{a} - \vec{a}_T.$$

The same formula in the coordinate form is follows:

$$\vec{a}_n = \frac{1}{v_x^2 + v_y^2}(a_x(v_x^2 + v_y^2)\vec{i} + a_y(v_x^2 + v_y^2)\vec{j} - (a_x v_x + a_y v_y)v_x\vec{i} - (a_x v_x + a_y v_y)v_y\vec{j}) = \frac{(a_x v_y - a_y v_x)}{v_x^2 + v_y^2}(v_y\vec{i} - v_x\vec{j})$$

The length of the normal vector:

$$|a_n| = \sqrt{(a, a) - (a_T, a_T)} = \frac{|a_x v_y - a_y v_x|}{|\vec{v}|}.$$

The curvature in a general case

The formula for curvature of the curve:

$$\rho = \frac{|a_n|}{(\vec{v},\vec{v})} = \frac{\sqrt{(a,a)-(a_T,a_T)}}{(\vec{v},\vec{v})} = \frac{|a_x v_y - a_y v_x|}{(\vec{v},\vec{v})^{3/2}} = \frac{|\vec{a}\times\vec{v}|}{(\vec{v},\vec{v})^{3/2}}.$$

The curvature. An example

Consider the flat motion with constant second derivative:

$$x(t) = v_1 t, \quad y(t) = at^2.$$

The tangent vector at t: $v_x = v_1, \quad v_y = 2at$. The instant second derivative: $a_x = 0, a_y = 2a$. The value of projection on the tangent direction is

$$|a_T| = \frac{(\vec{a}, \vec{v})}{\sqrt{(\vec{v}, \vec{v})}} = \frac{4a^2 t}{\sqrt{v1^2 + 4a^2 t^2}}.$$

The normal part of the second derivative:

$$|a_n| = \sqrt{4a^2 - \frac{16a^4 t^2}{v1^2 + 4a^2 t^2}} = \frac{2av_1}{\sqrt{v1^2 + 4a^2 t^2}}.$$

The curvature. An example

The curvature of the trajectory:

$$\rho = \frac{2av_1}{(v_1^2 + 4a^2t^2)^{3/2}}.$$

Therefore the maximum of the curvature is:

$$\rho_{max} = \frac{2a}{v_1^2},$$

and the curvature tends to zero as $t \to \infty$. The tangent projection is $|a_T| = \frac{4a^2t}{\sqrt{v1^2+4a^2t^2}} \to 2a$ as $t \to \infty$. The normal vector $\frac{2av_1}{\sqrt{v1^2+4a^2t^2}} \to 0$ as $t \to \infty$.

General formulas

The radius-vector for the trajectory is

$$\vec{r} = (x(t), y(t), z(t)).$$

The tangent vector to the curve is following:

$$\vec{v} = \frac{d}{dt}\vec{r} = (\dot{x}, \dot{y}, \dot{z}).$$

The second derivative is:

$$\vec{a} = \frac{d^2}{dt^2}\vec{r} = (\ddot{x}, \ddot{y}, \ddot{z}).$$

The vector of the second derivatives in 3D

The tangent projection of the vector of a second derivative:

$$\vec{a}_T = \frac{(\vec{a}, \vec{v})}{(\vec{v}, \vec{v})}\vec{v} = \frac{a_x v_x + a_y v_y + a_z v_z}{v_x^2 + v_y^2 + v_z^2}(v_x \vec{i} + v_y \vec{j} + v_z \vec{k}).$$

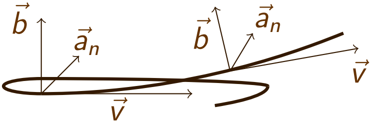
The normal component of the second derivative vector:

$$\vec{a}_n = \vec{a} - \vec{a}_T.$$

The normal and tangent vectors define the *osculating plane*. Define a unit vectors $\vec{u} = \frac{\vec{v}}{\sqrt{(\vec{v}, \vec{v})}}$ and $\vec{n} = \frac{\vec{a}_n}{\sqrt{(\vec{a}_n, \vec{a}_n)}}$. The vector $\vec{b} = \vec{u} \times \vec{n}$ is called *binormal*. The vectors $\vec{u}, \vec{n}, \vec{b}$ define the orthogonal system of the vectors connected with the curve.

A torsion of the curve

Torsion is a derivative of the angle of rotation of osculating plane with respect to changing the length of the curve.



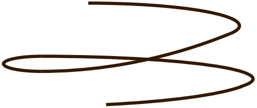
The normal vector to the osculation plane:

$$\vec{b} = \vec{v} \times \vec{a}.$$

The formula for the torsion has the form:

$$\tau = |\dot{\vec{b}}| \frac{dt}{dl}.$$

An example helix.

Helix: 

$$\vec{r} = (cos(t), \sin(t), t), \quad \vec{v} = (-\sin(t), \cos(t), 1)$$

$$\vec{a} = (-\cos(t), -\sin(t), 0); \quad \vec{b} = \vec{v} \times \vec{a} = (-\sin(t), -\cos(t), 0).$$

$$\dot{\vec{b}} = (-\cos(t), \sin(t), 0), \quad dl = \sqrt{\sin^2(t) + \cos^2(t) + 1 dt}.$$

Torsion:

$$\tau = |\dot{\vec{b}}| \frac{dt}{dl} = \frac{\sqrt{\cos^2(t) + \sin^2(t)}}{\sqrt{\sin^2(t) + \cos^2(t) + 1}} = \frac{1}{\sqrt{2}}.$$

Summary

- A length of a path is defined curvilinear integral of the second kind.
- Value of normal acceleration and speed define the curvature of a curve.
- Binormal vector define the osculating plane.
- Torsion define the change of the binormal vector along the trajectory.