Lecture 2 (part 2)

Functions and limits

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A variable.

In practice we use a lot of quantities which vary and change their value.

- a time, say „t";
- a distance, say „x";
- temperature, say „T";

Such quantities we define as „variables".

In calculus we neglect a physical meaning of those quantities.

Instead them we use word „variable".

The variabl can obtain values from some set.

An variable

we will assume that the variable is defined, if we indicate a letter for the variable and a set of values for the variable:

- $x \in \mathbb{R}$ (the variable „x" can obtain value in set of real numbers) shortly: „x" is real;
- $n \in \mathbb{N}$, „n" is natural;
- $q \in \mathbb{Q}$, „q" is rational;
- $x \in \mathbb{R}$, $x > 0$: „x" is positive real
- $y \in [a, b]$ „y" belongs to the interval a, b.

Intervals

$[a, b] = \{x \in \mathbb{R},\ a \leq x \leq b\};$

$(a, b] = \{x \in \mathbb{R},\ a < x \leq b\};$

$[a, b) = \{x \in \mathbb{R},\ a \leq x < b\};$

$(a, b) = \{x \in \mathbb{R},\ a < x < b\}.$

Minimum of x on $[a, b]$ is „a"
maximum of x on $[a, b]$ is „b"

The open interval (a, b) doesn't have minimal and maximal value of x.

„a" is **infimum** of x on (a, b).
„b" is **supremum** of x on (a, b).

Functions.

Nature gives us a lot of dependencies:

- an outdoor temperature depends on season.
- the highest position of the Sun depends on the altitude of a place;
- a pressure in a tyre depends on a temperature

Such dependencies we will consider as functional dependencies neglecting their physical sense.

$y(x)$ - „y" depends on „x", or

„y" is a function of „x".

Functions. Examples.

$y = 2x + 3$ – „y" is the linear fuction of „x";

$y = 2x^2 + 4x + 2$ – „y" is the quadratic function of „x";

$y = \frac{5x-6}{x+1}$ – „y" is the rational function of x

Def. A set of all real numbers for which a certain function is defined we will call a implied domain.

Def. A set of all values of a certain function we will call a range of a function

Functions. Examples.

1) $y = \sqrt{x^2 - 1}$, the implied domain $x \in (-\infty, -1] \cup [1, +\infty)$
or the same: $x \in \mathbb{R} \setminus (-1, 1)$.
the range of y: $y \in [0, +\infty)$.

2) $y = \dfrac{\sqrt{x+2}}{x-1}$, $x \in [-2, 1) \cup (1, +\infty)$, or
$x \in [-2, +\infty) \setminus 1$.
the range of y: $y \in (-\infty, +\infty)$.

Def. If both border points are included into an interval, we will write $[a, b]$ (a closed interval).

Def. If border point, say „a", is excluded from the interval we will write $(a, b]$. (an left-open, right-closed interval)

Explicit forms of functions

1) $y = \sin(x)$

2) $y = |x|$

3) $y = \mathrm{sign}(x) = \begin{cases} 1, & x > 0; \\ 0, & x = 0; \\ -1, & x < 0, \end{cases}$

4) $y = \sqrt{9 - x^2}$

5) $y = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$ The Dirichlet's function

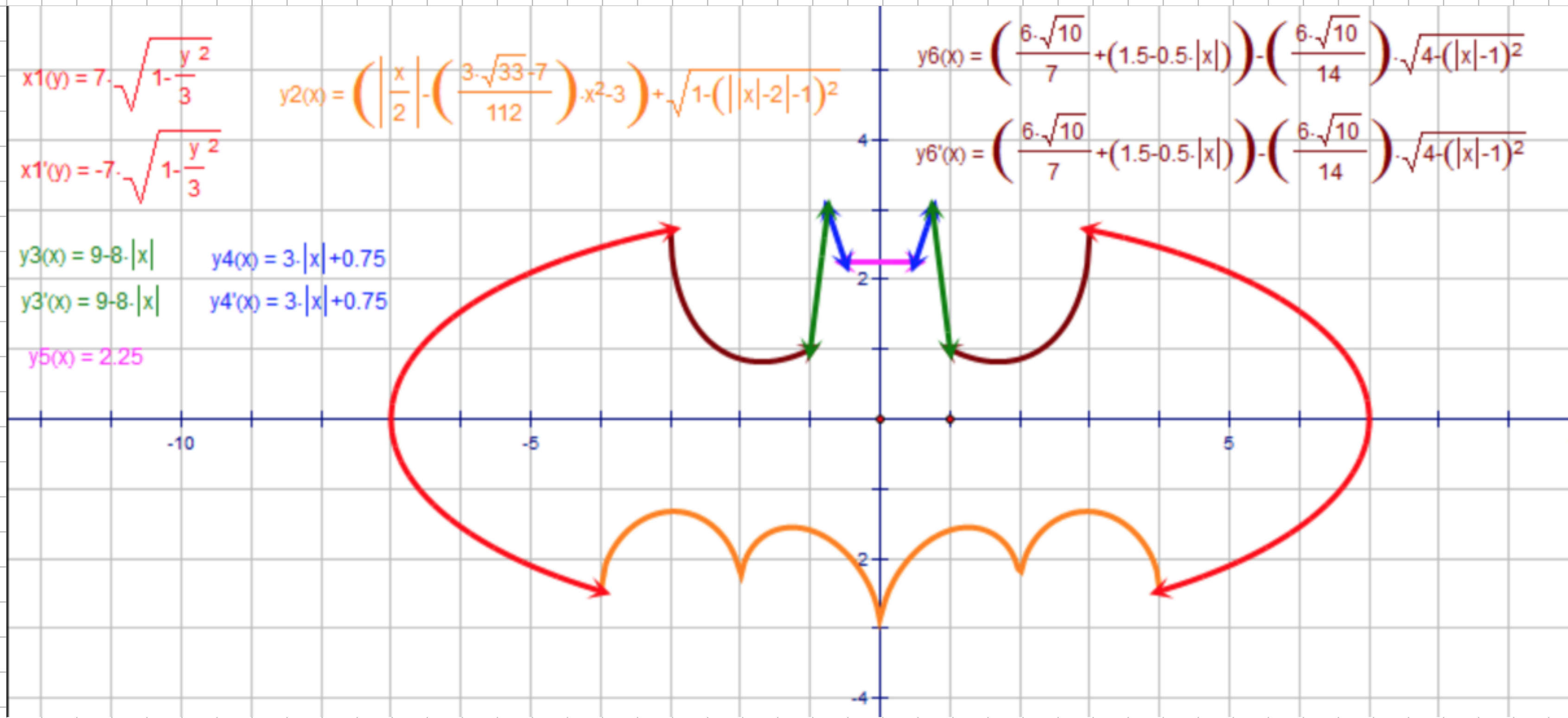
Implicit forms of functions and multivalued functions.

1) $y^2-|x|=0 \Rightarrow y=\begin{cases}\sqrt{x}, & x\geq 0, \ y\geq 0;\\ \sqrt{-x}, & x<0, \ y\geq 0;\\ -\sqrt{x}, & x\geq 0, \ y<0;\\ -\sqrt{-x}, & x<0, \ y<0.\end{cases}$

2) $x^2+y^2=4 \qquad y=\begin{cases}\sqrt{4-x^2}, & y\geq 0;\\ -\sqrt{4-x^2}, & y<0.\end{cases}$

3) $x^2-y^2=4. \qquad y=\begin{cases}\sqrt{x^2-4} & y\geq 0,\\ -\sqrt{x^2-4} & , y<0.\end{cases}$

Stolen from the Internet.



Thank you, an unknown Author!

Definition of a limit.

A number „a" is called a limit of a variable x if $\forall \varepsilon > 0 \ \exists \ x : |a - x| < \varepsilon$. We will write:

$$x \to a \quad \text{or} \quad \lim x = a.$$

Examples:

1) $x \in \mathbb{Q}$ and $x = 1 + \frac{1}{2}, x = 1 + \frac{1}{3}, \ldots x = 1 + \frac{1}{n}, \ldots \lim x = 1$

2) $x \in \{0, 1\}$ and $x = 0, x = 1, \ldots 0, \ldots 1$ Then x has no lim

Let $f(x)$ defined on $x \in (a, b)$, exept possible a point x_0, $x_0 \in (a, b)$. A number A is called a limit of $f(x)$ as $x \to x_0$ if

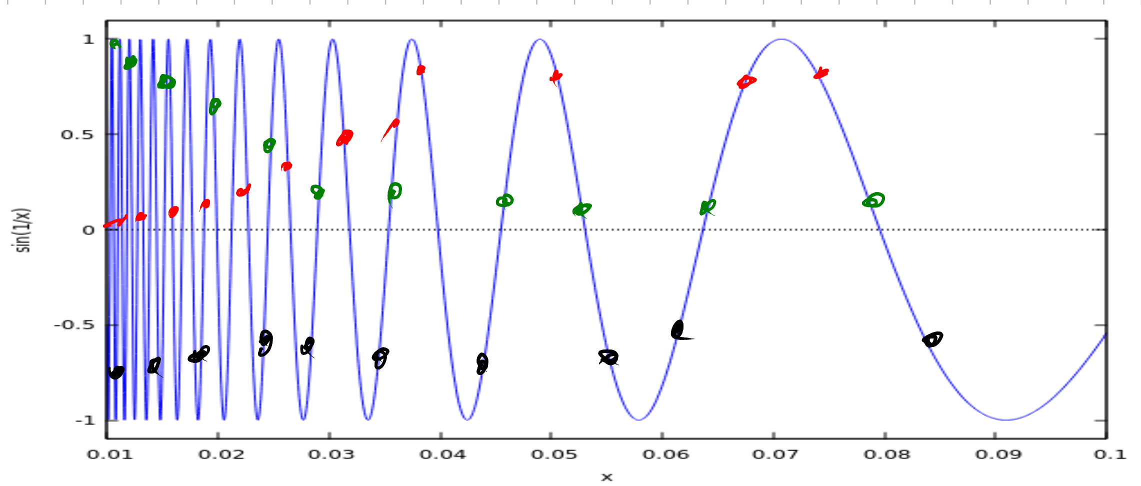
$$\forall \{x_n\}, \quad \lim_{n\to\infty} x_n = x_0, \quad f_n = f(x_n) \quad \lim_{n\to\infty} f_n = A.$$

Examples:

1) $f(x) = x^2$, $\forall \{x_n\}$ $x_n \to 0$ $f_n \to 0$ $\lim\limits_{x\to 0} f(x) = 0$

2) $f(x) = \sin\left(\frac{1}{x}\right)$

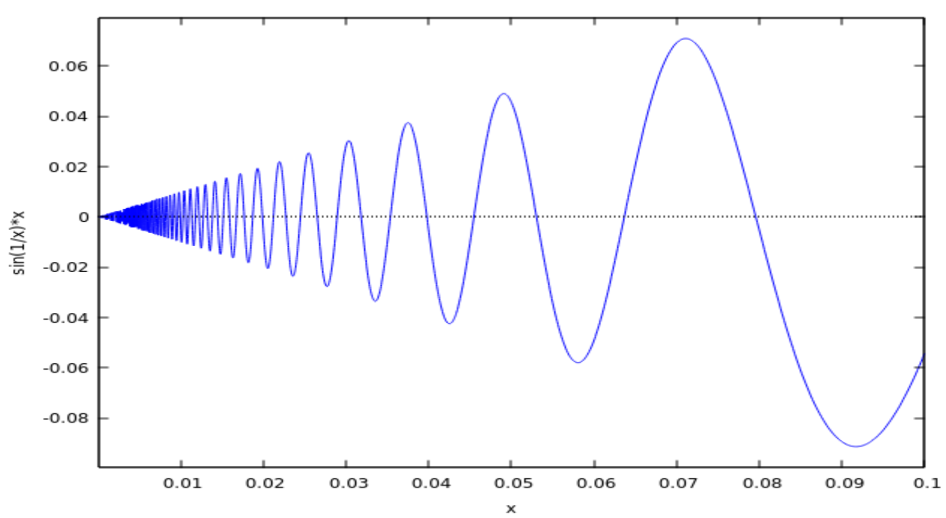
The function has no a limit as $x \to 0$.



3) $f(x) = x \sin\left(\frac{1}{x}\right)$

The function has limit as $x \to 0$:

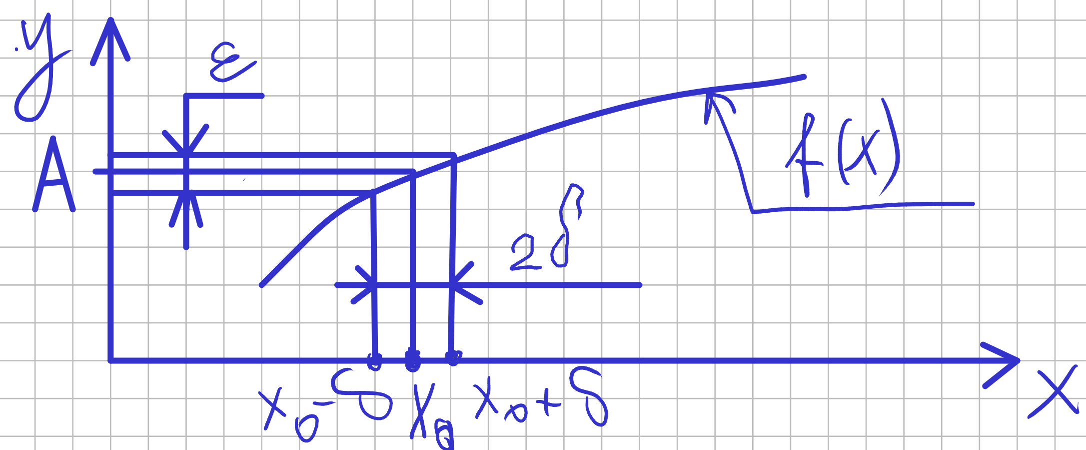
$$\lim_{x \to 0} f(x) = 0.$$



Second definition of a limit.

Let $f(x)$ defined on $x \in (a,b)$, exept possible a point x_0, $x_0 \in (a,b)$. A number A is called a limit of $f(x)$ as $x \to x_0$ or $\lim\limits_{x \to x_0} f(x) = A$ if.

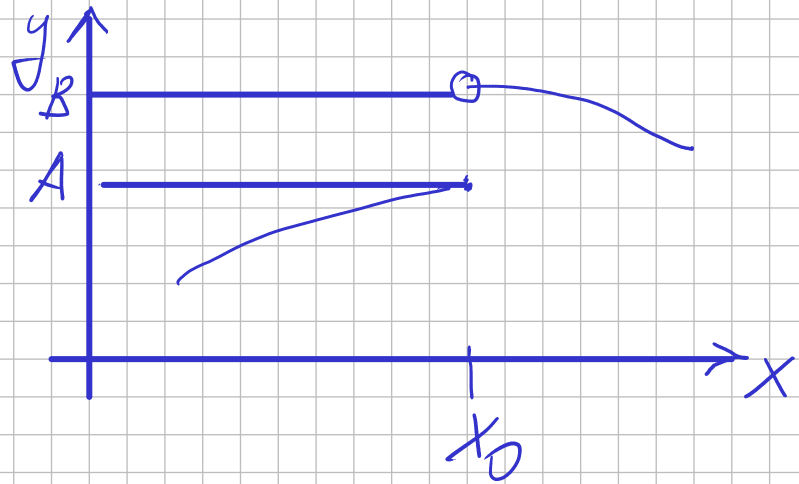
$$\forall \varepsilon > 0 \ \exists \delta(\varepsilon) > 0 : |x - x_0| < \delta : |f(x) - A| < \varepsilon$$



One side limits

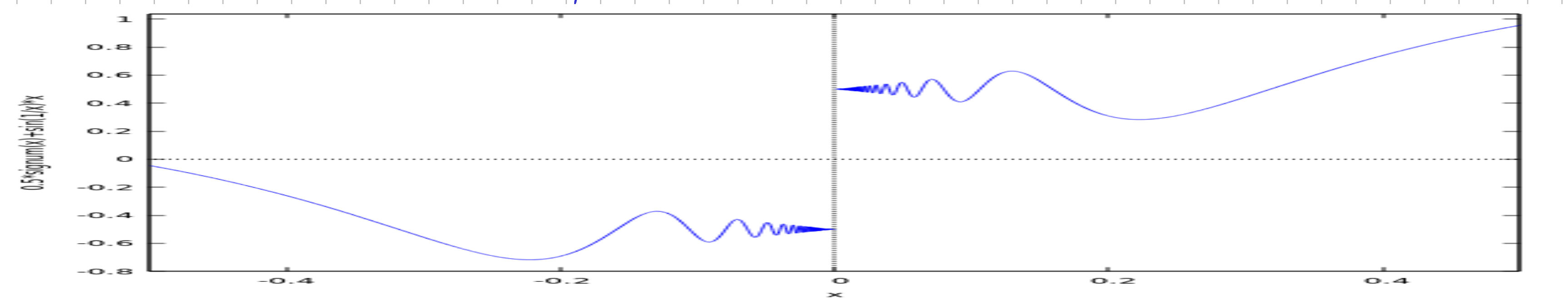
$$\lim_{x\to x_0-0} f(x) = A$$

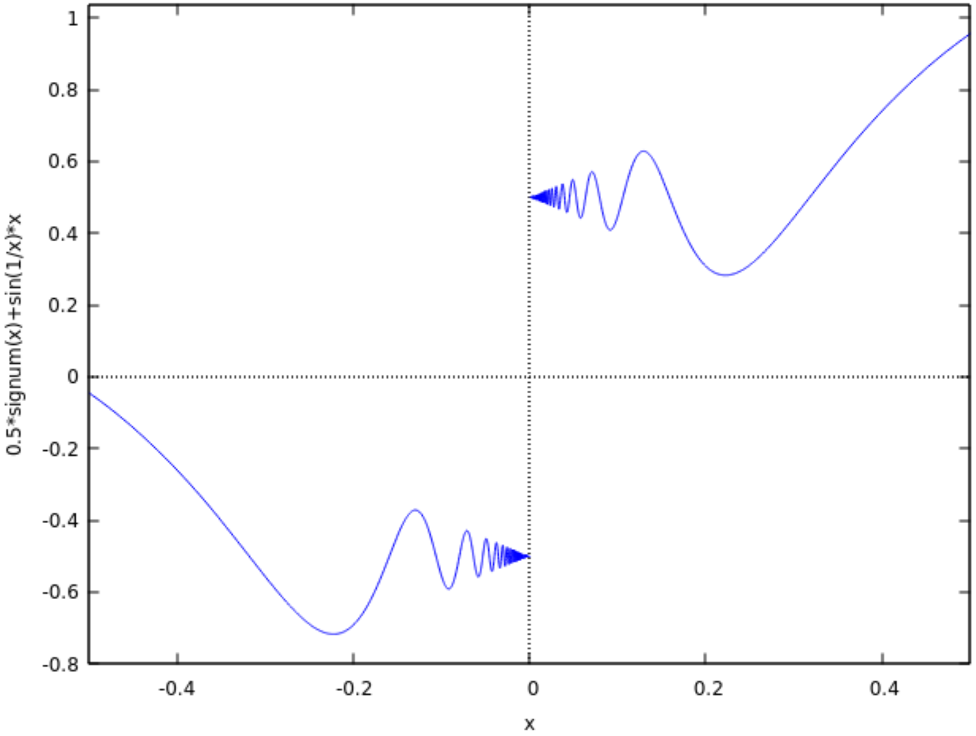
$$\lim_{x\to x_0+0} f(x) = B$$



1) $\forall \varepsilon>0\ \exists\, \delta(x)>0,\ \underline{\underline{x-\delta<x<x_0}}:\ |f(x)-A|<\varepsilon.$

2) $\forall \varepsilon>0\ \exists\, \delta(x)>0,\ \underline{\underline{x_0<x<x+\delta}}:\ |f(x)-B|<\varepsilon.$





$$f(x) = \frac{1}{2}\,\mathrm{sign}(x) + x\sin\left(\frac{1}{x}\right)$$

$x_0 = 0$ does not belong to the implied domain

but one side limits exist!

$$\lim_{x\to -0} \frac{1}{2}\,\mathrm{sign}(x) + x\sin\left(\frac{1}{x}\right) = -\frac{1}{2}, \quad \lim_{x\to +0} \frac{1}{2}\,\mathrm{sign}(x) + x\sin\left(\frac{1}{x}\right) = \frac{1}{2}$$

Properties of limits.

1) if $\lim\limits_{x\to x_0} f(x) = A$, $\lim\limits_{x\to x_0} h(x) = B \Rightarrow \lim\limits_{x\to x_0}\left(f(x)+h(x)\right) = \lim\limits_{x\to x_0} f(x) + \lim\limits_{x\to x_0} h(x)$.

2) if $\lim\limits_{x\to x_0} f(x) = A$, $\lim\limits_{x\to x_0} h(x) = B \Rightarrow \lim\limits_{x\to x_0}\left(f(x)h(x)\right) = \lim\limits_{x\to x_0} f(x) \cdot \lim\limits_{x\to x_0} h(x)$.

Both properties are corollaries of the same properties for sequences and the definition of the limit as a limit of $\forall$ sequence.

But straightforward proof can be done using second definition of the limit.

The rule of a change of variables

Theorem if $\exists \lim\limits_{x \to x_0} f(x) = y_0$, and $f(x) \neq y_0$ as $x \neq x_0$,

and $\exists \lim\limits_{y \to y_0} F(y)$, then as $x \to x_0$ $\exists \lim\limits_{x \to x_0} F(f(x)) = y_0$

Proof: $\forall \{x_n\} \to x_0$ define $f_n = f(x_n) \Rightarrow \{f_n\} \to y_0 \Rightarrow$

$F_n = F(f_n)$: $\{F_n\} \to y_0$, $n \to \infty$

Examples

1) $f(x) = x^2+\pi$, $F(y) = \sin(y)$, $\lim\limits_{x\to 0} \sin(x^2+\pi) = 0$.

Counterexample

2) $\lim\limits_{x\to 0} \dfrac{\sin(x\sin(\frac{1}{x}))}{x\sin(\frac{1}{x})}$, then $F(y) = \dfrac{\sin(y)}{y}$ and $f(x) = x\sin(\frac{1}{x})$,

$F(0)$ — does not defined, but:

$x \cdot \sin(\frac{1}{x}) = 0$ as $x \neq 0$; $\sin(\frac{1}{x}) = 0 \Rightarrow \frac{1}{x} = \pi n \Rightarrow x_n = \frac{1}{\pi n}$

$\forall \varepsilon > 0$ $\exists N$, $n > N$ $x_n \sin(\frac{1}{x_n}) = 0 \Rightarrow \lim$ does not exist!

Theorem about monotonic functions.

If $f(x)$ monotonously increases on (a,b),

then $\lim\limits_{x\to b-0} f(x) = \sup\limits_{x\in(a,b)} f(x)$.

and vice versa

if $f(x)$ monotonously decreases on (a,b)

then $\lim\limits_{x\to b-0} f(x) = \inf\limits_{x\in(a,b)} f(x)$

Proof. The stament is straightforward corollary the theorem for monotonic sequence!

Summary

1) Two definitions of limit of function
2) One side limits
3) Properties of limits
4) Theorem: a rule of a change variables.
5) limits of monotonic functions.