Lecture 1 (part 2)

Sequences,

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Sequences.

In Calculus we will often use sequences $x=a_1$, $x=a_2$, $x=a_3$, ... and so on.

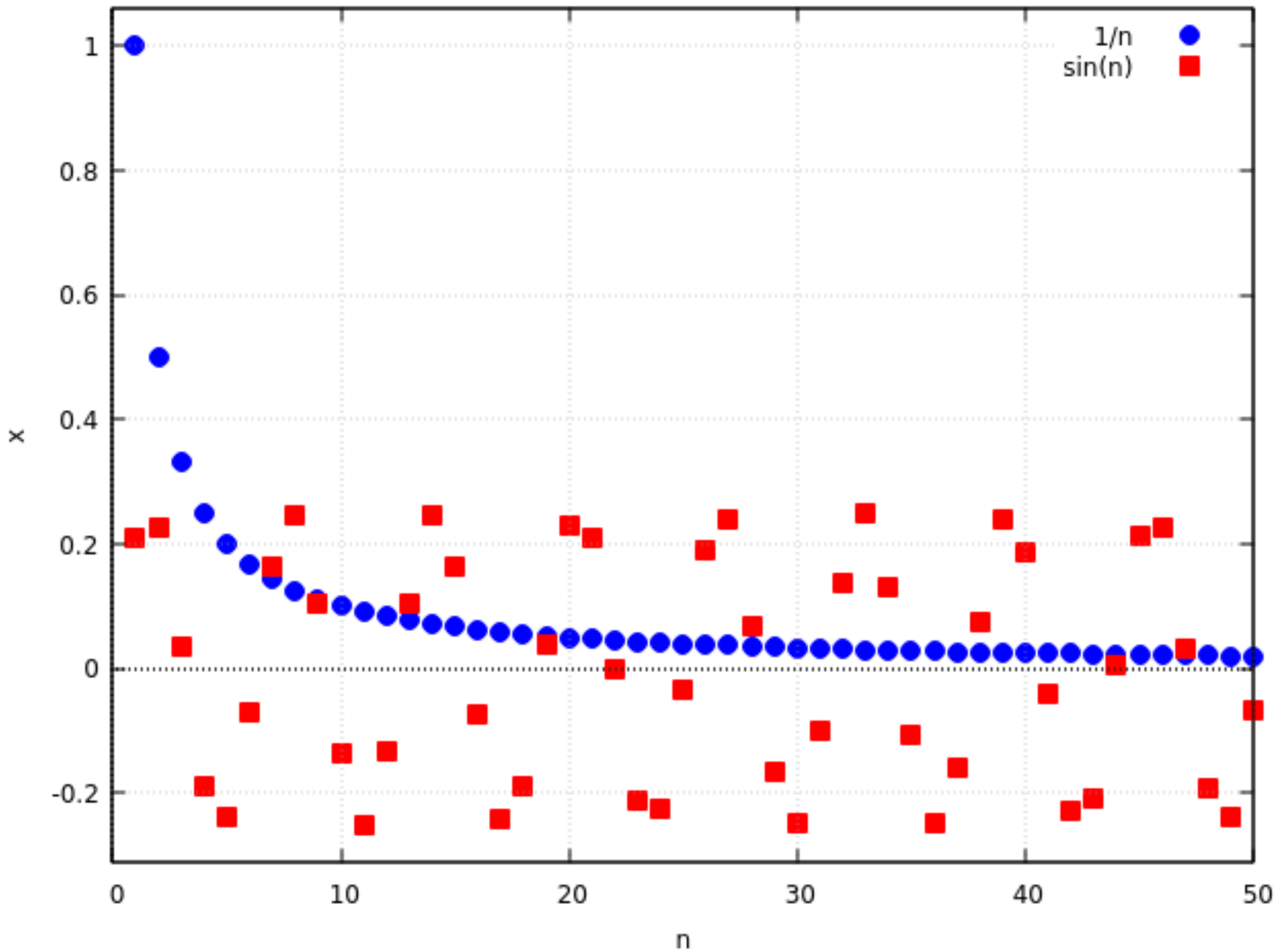
Def. The formula $\{a_n\}_{n=1}^{\infty}$ means infinite sequence of constants. Mostly we will assume $a_n \in \mathbb{R}$, $\forall n \in \mathbb{N}$.

Def. We say the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded if $\exists A \in \mathbb{R}$, $\forall n \in \mathbb{N}$ $|a_n| < A$.

Example. 1) $a_n = \sin(n)$, $n \in \mathbb{N}$.

2) $a_n = \frac{1}{n}$, $n \in \mathbb{N}$.

~~3) $a_n = n^2$, $n \in \mathbb{N}$~~



- The convergent sequence $x_n = 1/n$.
- The bounded sequence $x_n = \frac{1}{4}\sin(n)$.

The last one $a_n = n^2$, $n \in \mathbb{N}$ is unbounded or divergent sequence

Conversely the sequence $a_n = \frac{1}{n}$, $n \in \mathbb{N}$ is convergent.

For formal definition such constructions we need a defintion of limit.

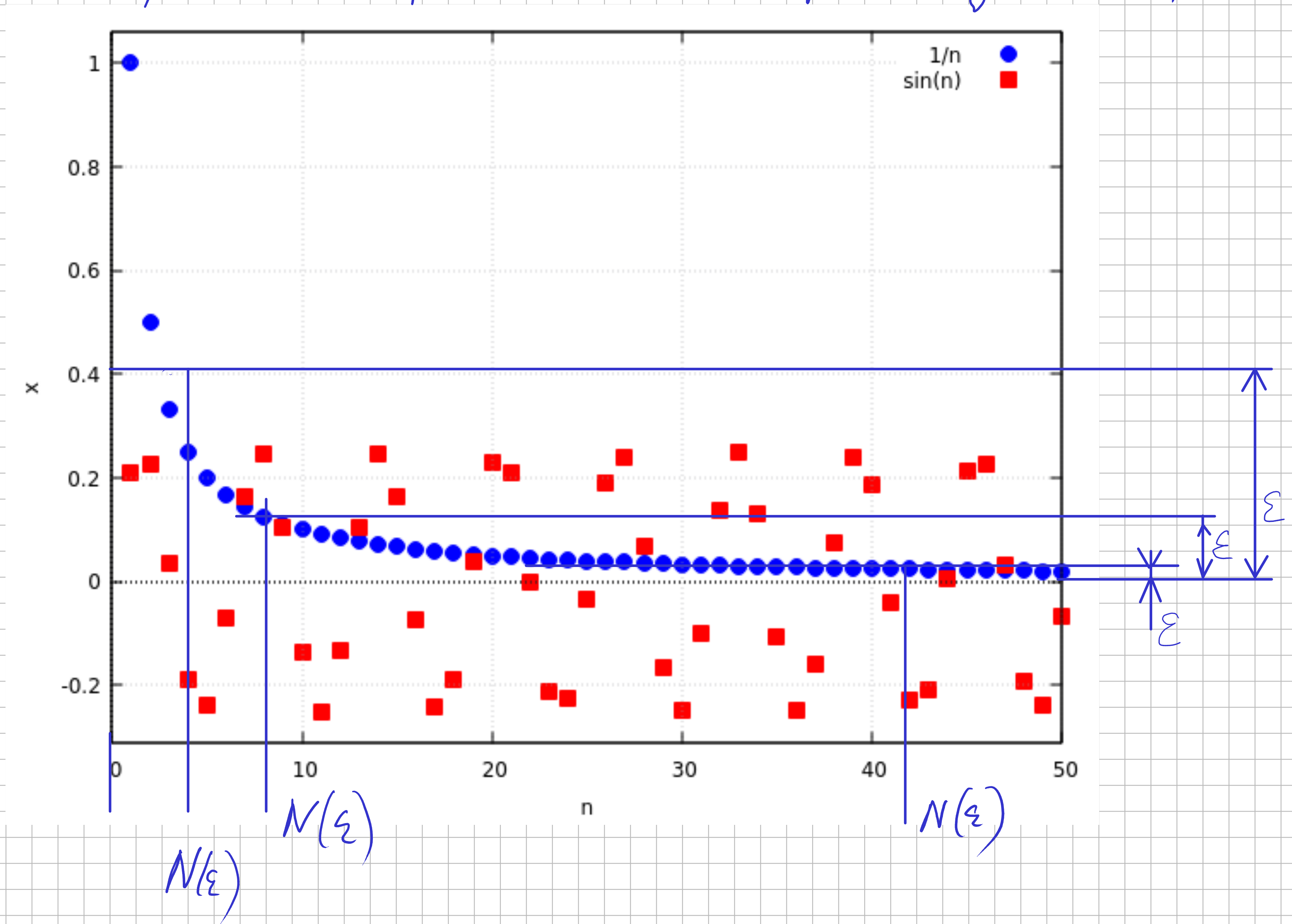
Def. A number A is a limit of $\{a_n\}_{n=1}^{\infty}$ if $\forall \varepsilon > 0 \ \exists N$ such that $\forall \ n > N$: $|A - a_n| < \varepsilon$.

$$\lim_{n\to\infty} a_n = A \quad \text{or:} \quad a_n \to A, \ n \to \infty.$$

The same definition in math logic notation:

$$\lim_{n\to\infty} a_n = A \iff (\forall \varepsilon > 0)(\exists N \in \mathbb{N})(n > N) : |A - a_n| < \varepsilon.$$

An example for explanation of dependency $N(\varepsilon)$ in the definition of the limit of a sequence.



Def A sequence is called unbounded if

$$\lim_{n\to\infty}|a_n| = \infty \iff (\forall M\in\mathbb{R})(\exists N\in\mathbb{N})(\forall n>N): |a_n|>M$$

Def. A sequence is called convergent if

$$\forall \varepsilon>0 \ \exists N\in\mathbb{N} \ \forall n,m>N : |a_n-a_m|<\varepsilon$$

Example: $\{x_n\}_{n=1}^{\infty}$, $x_n\in\mathbb{Q}$:

$$x_{n+1}=\frac{1}{2}\left(x_n+\frac{2}{x_n}\right), \quad x_1 = 1{,}1.$$

$x_n \to \sqrt{2}$, But $\sqrt{2}\notin\mathbb{Q} \Rightarrow \{x_n\}_{n=1}^{\infty}$ converges in $\mathbb{Q}$ as $n\to\infty$, but this sequence does not have any limit in $\mathbb{Q}$

Not all convergences have a limit!

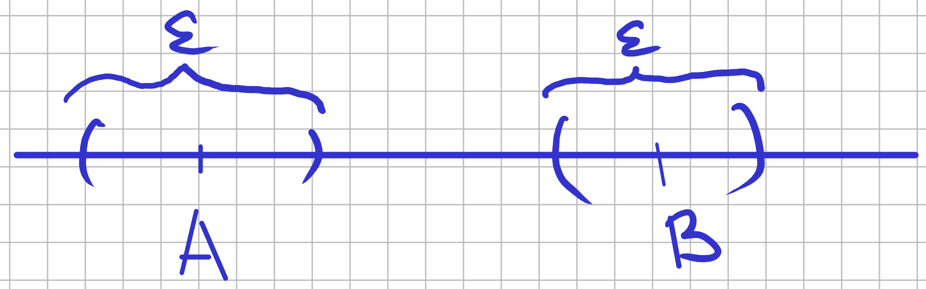
Theorem. A sequence might have only one limit!

Proof. Let's suppose there exist two different limits A and B and $A < B$

then $\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n > N \ |A - x_n| < \varepsilon$

and $\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n > N \ |B - x_n| < \varepsilon$

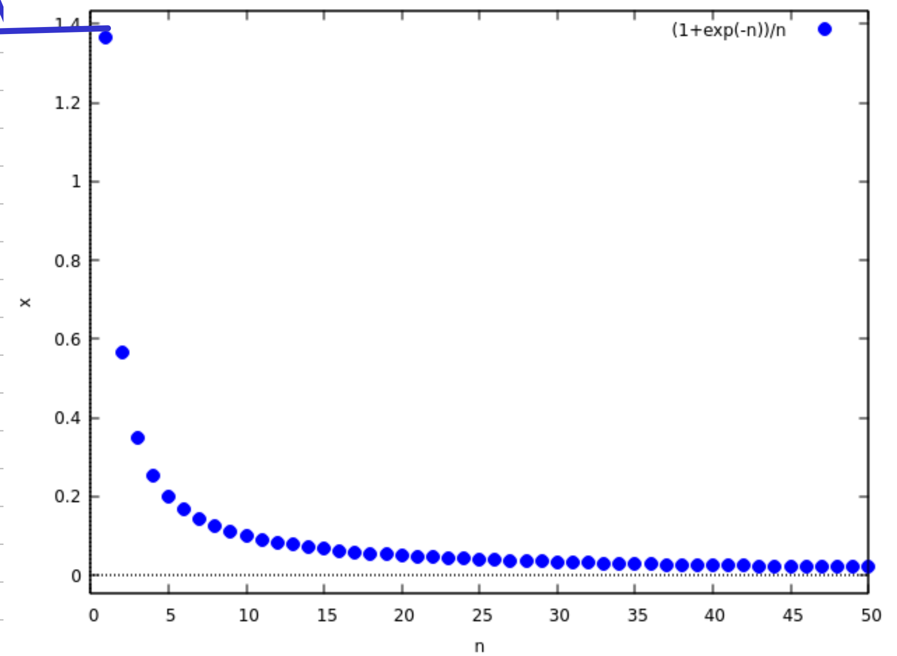
Consider $\varepsilon < B - A \Rightarrow$ we obtain contradiction!



$\inf\{a_n\}$ is $\max\limits_{b\in\mathbb{R}}\{\forall n, b < a_n\}$,

$\sup\{a_n\}$ is $\min\limits_{b\in\mathbb{R}}\{\forall n\ b > a_n\}$.

$\sup\{a_n\}$



$\inf\{a_n\}$

Example $a_n = (1+e^{-n})\cdot\frac{1}{n}$, $n \in \mathbb{N}$

$\inf\{a_n\} = 0$ $\quad$ $\sup\{a_n\} = 1+e^{-1}$

Def A sequence grows monotonously if $\forall n: a_{n+1} > a_n$

Theorem. Any bounded monotonously growing sequence has a limit.

Proof. As well as the sequence is bounded then $\exists\, b = \sup\{a_n\}$. The sequence grows, therefore $\forall \varepsilon > 0\ \exists\, N \in \mathbb{N},\ n > N \quad b - \varepsilon < a_n < a_{n+1} < \ldots < b$

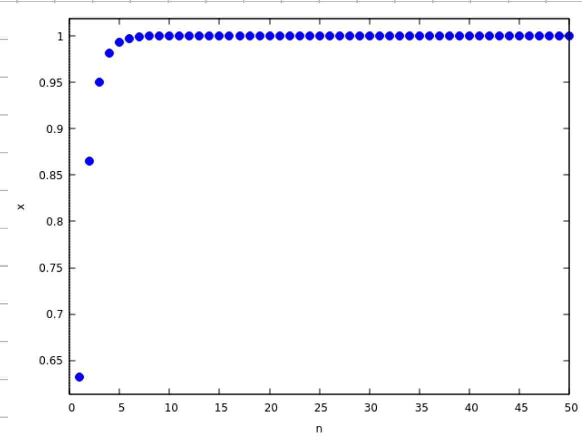
$\Rightarrow b = \lim\limits_{n \to \infty} a_n$.

Example 1. $a_n = 1 - e^{-n}$ This sequence grows monotonously

Example 2. $a_n = 1 - e^{-n} \cos(n)$ This sequence does not grow monotonously.

Example 1) $a_n = (1-e^{-n})$

$$\lim_{n\to\infty} a_n = 1$$



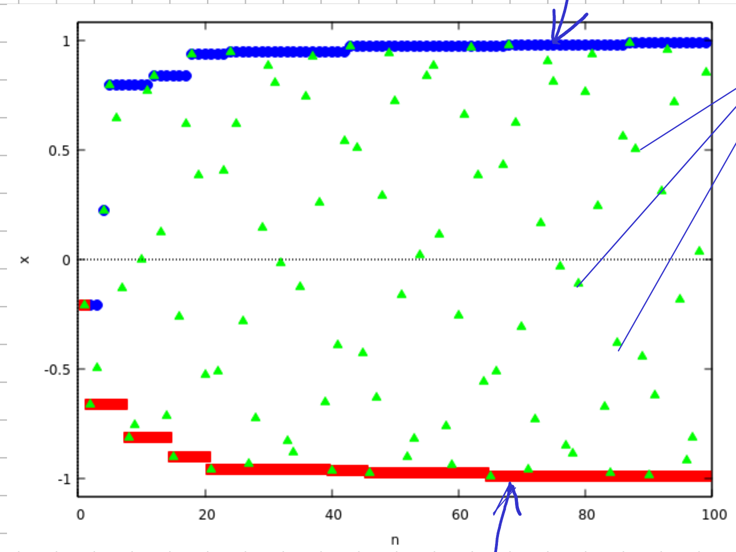
2) $a_n = \left(1-\frac{1}{n}\right)\cos(n)$

$$b_m = \max\{a_n\}_{n=2}^{m}$$

$$\lim_{m\to\infty} b_m = 1$$

b_n

$\left(1-\frac{1}{n}\right)\cos(n)$



c_n

3) $c_m = \min\{a_n\}_{n=2}^{m}$

$$\lim_{m\to\infty} c_n = -1$$

Show that $\forall \varepsilon>0 \ \exists n : |\cos(n)-1|<\varepsilon$

$n - m\pi \to 0$

$\frac{n}{m} - \pi \to 0$

Def. $\{b_k\}_{k=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$ if $\forall k\ \exists m:\ b_k = a_{m_k}$, and if $k_1 < k_2$ then $m_{k_1} < m_{k_2}$.

Theorem (Bolzano - Weirstrass) $\forall$ bounded $\{a\}_{n=1}^{\infty}$ contains convergent subsequence

Proof. $a = \inf\{a_n\}$, $b = \sup\{a_n\}$, consider $[a, \frac{a+b}{2}]$ and $(\frac{a+b}{2}, b]$, then one of those intervals contains subsequence. Repeating this process we obtain convergent sequence of intervals which contains subsequence. Those consequence convergent due to convergency of the intervals.

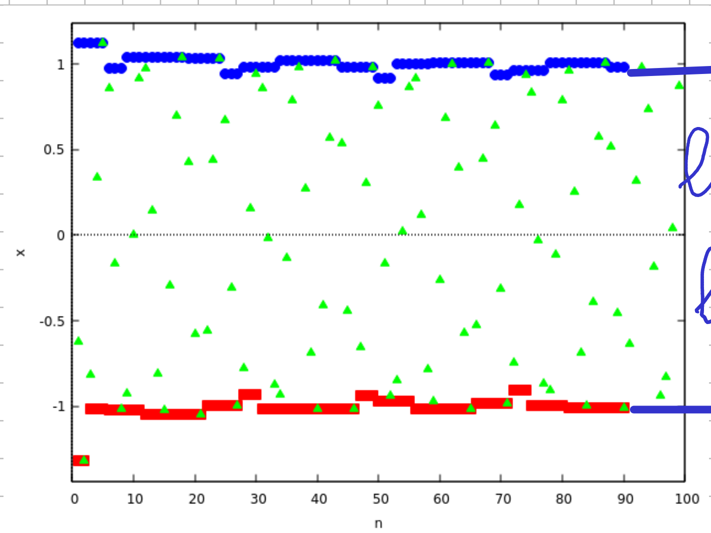
Def. | A limit superior is a maximal of subsequental limits

Def. | A limit inferior is a minimal of subsequental limits

<u>Example</u> $a_n = \left(1 - \frac{1}{n}\right)\cos(n)$

$\overline{\lim\limits_{n\to\infty}}\, a_n = 1$ or $\limsup\limits_{n\to\infty} a_n = 1$

$\underline{\lim\limits_{n\to\infty}}\, a_n = -1$ $\liminf\limits_{n\to\infty} a_n = -1$.



$\limsup(a_n)$

$\liminf(a_n)$

Summary

1. A sequence
2. A limit of a sequence
3. Convergent sequences
4. Properties of monotonous sequences
5. Bounded sequences and subsequences.
6. $\overline{\lim} \equiv \limsup$, $\underline{\lim} \equiv \liminf$.