Lecture 1 (part 1)

Real digits.

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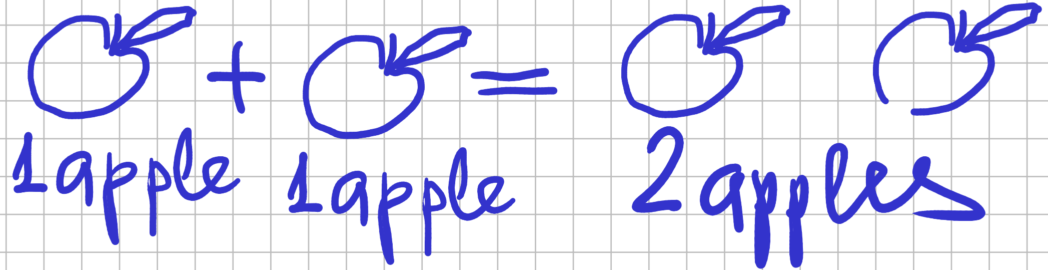
Natural numbers:



Step 1. Make abstract objects:

$$\frac{\text{1 apple}}{\text{1 apple}} = \frac{1}{1} = 1.$$

The set of natural numbers $\mathbb{N} = \{1, 2, 3, 4, \dots\}$

Step 2. Make a semigroup:

$\mathbb{N}$ and binary operator: addition

Problem: we do not have an inverse operator

$$a + x = b$$
$$2 + x = 1$$

we cannot solve the equations

Integer numbers: $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-1, -2, \dots\}$.

Properties: $\mathbb{Z}$ and $+$ $\Rightarrow$ abelian group:

1. $a, b \in \mathbb{Z}$: $a+b = b+a$;
2. $a+0 = a$
3. $\forall a \in \mathbb{Z}\ \exists c$: $a+c = 0$

Multiplication:
1. $a, b \in \mathbb{Z} \Rightarrow a \cdot b = b \cdot a \in \mathbb{Z}$
2. $a \cdot 1 = a$

Problem: We do not have an inverse operation (division)

We cannot solve the equation:
$ax + b = 0$;
$3x + 1 = 0$

Rational numbers $\mathbb{Q}$: $m \in \mathbb{N}$, $n \in \mathbb{Z}$, $q = \frac{m}{n}$

① $p+q = \frac{k}{l} + \frac{m}{n} = \frac{k\cdot n + l\cdot m}{l\cdot n}$, $k\cdot n + l\cdot n \in \mathbb{Z}$, $l\cdot n \in \mathbb{N}$

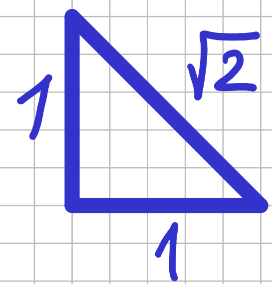
$p+q \in \mathbb{Q}$.

② $p\cdot q = \frac{k}{l}\cdot\frac{m}{n} = \frac{k\cdot m}{l\cdot n} \in \mathbb{Q}$

We can solve a linear equations:

$ax + b = 0$, $a \in \mathbb{Q}$, $b \in \mathbb{Q}$

if $a \neq 0$. $x = \frac{-b}{a} \in \mathbb{Q}$



Problem: What can we say about quadratic equations?

$$x^2 - 2 = 0$$
$$x = \pm\sqrt{2}$$

A quadratic equation: $x^2 - 2 = 0$

Let's define a symbol $\sqrt{2}$ as a solution

Whether $\sqrt{2} \in \mathbb{Q}$, then $\exists p \in \mathbb{Z}$ and $q \in \mathbb{N}$, such that $\sqrt{2} = \frac{p}{q}$

$$2 = \frac{p^2}{q^2},$$

$p = p_1^{l_1} \cdot p_2^{l_2} \cdot p_3^{l_3} \cdot \ldots \cdot p_n^{l_n}$, p_i is a prime number,

$q = q_1^{m_1} \cdot q_2^{m_2} \cdot q_3^{m_3} \cdot \ldots \cdot q_k^{m_k}$, q_i is a prime number.

$$2q^2 = p^2$$

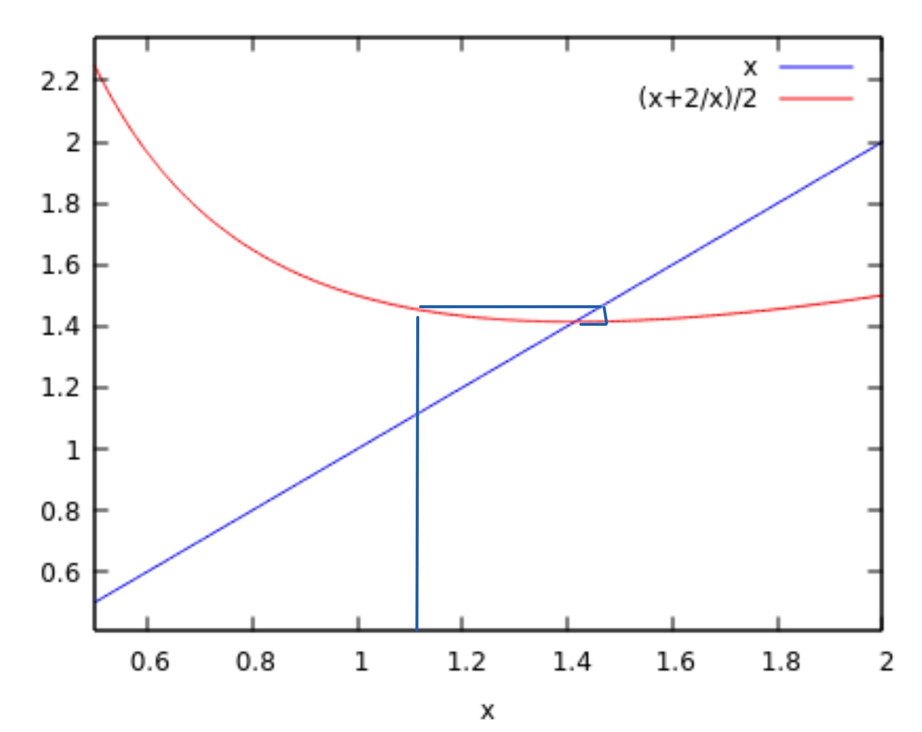
$\leftarrow$ even number of „2"

odd number of „2"

We obtain a contradiction!

An approximation procedure.

$$x^2 = 2$$
$$x^2 + x^2 = 2 + x^2$$
$$2x^2 = x^2 + 2$$
$$x^2 = \frac{1}{2}(x^2+2)$$
$$x = \frac{1}{2x}(x^2+2)$$
$$x = \frac{1}{2}\left(x + \frac{2}{x}\right)$$



$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{2}{x_k}\right);$$

$x_1 = 1.1$, $x_2 = 1,459(09)$

$x_3 = 1,149\,037$

$x_4 = 1,14213\ldots.$ $x_4^2 \approx 2,00000\ldots$

it's a Magic!

An approximation procedure:

We obtain a sequence of rational numbers $\{x_k\}_{k=1}^{\infty}$, $x_k \in \mathbb{Q}$, such that when $k \to \infty$

1. $x_{k+1} - x_k \to 0$, and
2. $\boxed{x_k \to \sqrt{2}, \text{ but } \sqrt{2} \notin \mathbb{Q}}$

Rational numbers are densely ordered

Proposition. For $\forall\, p, q \in \mathbb{Q}$, $p < q$
$\exists\, z \in \mathbb{Q}$: $p < z < q$.

Proof: $p = \frac{m}{n}$, $q = \frac{k}{l}$.

$$\frac{m}{n} < \frac{m+k}{n+l} < \frac{k}{l}$$

Explain!

Rationals are countable!

1	2	3	4	5	6	7
1/2	$\cancel{2/2}$	3/2	$\cancel{4/2}$	5/2	6/2	7/2
1/3	2/3	$\cancel{3/3}$	4/3	5/3	6/3	7/3
1/4	$\cancel{2/4}$	3/4	4/4	5/4	6/4	7/4
1/5	2/5	3/5	4/5	5/5	6/5	7/5

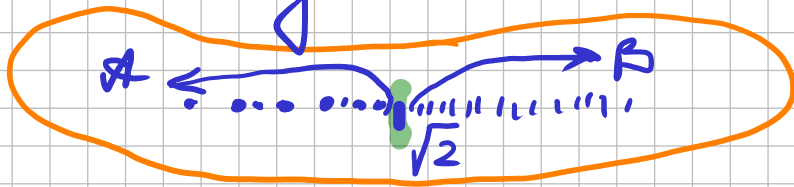
Problem:
How to get an uncountable set of numbers and associate such new set to a straight line?

This way allows us to get one-to-one correspondence between rational and natural numbers.

Let's cut the set $\mathbb{Q}$ by number $\sqrt{2}$:

$\forall a \in A: \quad a < \sqrt{2},$

$\forall b \in B: \quad b > \sqrt{2}.$



This way defines the number $\sqrt{2}$ using two sets of rationals $A \subset \mathbb{Q}$ and $B \subset \mathbb{Q}$

$A \cap B = \emptyset, \quad A \cup B = \mathbb{Q}.$

another way:

$(A, B) \equiv \sqrt{2}$ if $\forall a \in A \subset \mathbb{Q}, \; a < \sqrt{2}$, and

$\forall b \in B \subset \mathbb{Q}, \; b > \sqrt{2}.$

Note. The set B is addition of $\mathbb{Q}$ with respect to A, therefore, we can omit mentions of the set B.

Dedekind cuts.

Desect rational numbers on two nonempty subsets

$A \subset \mathbb{Q}$ and $B \subset \mathbb{Q}$, $A \cap B = \emptyset$, $A \cup B = \mathbb{Q}$.

$\forall a \in A, \forall b \in B: a < b$.

The digit which defines this desection we will call

Dedekind cut or (A, B) ← its only definition that the cut is defined by two subsets of $\mathbb{Q}$ which are A and B

There are following cases:

1) $\exists a_0 \in A, \forall a \in A \mid a \neq a_0 \quad a < a_0$
2) $\exists b_0 \in B, \forall b \in B \mid b \neq b_0 \quad b > b_0$
3) $z \notin A, z \notin B, \forall a \in A \quad z > a, \forall b \in B \quad z < b$.

Consider two cuts: (A,B) and (C,D), $(A,B) < (C,D)$ if $A \subset C$.

So we define the order for the cuts.

So we fill all gaps between the rationals.

Define a set of real numbers $\mathbb{R}$ as a union of all cuts of rational numbers.

Proposition. $\forall \alpha, \beta \in \mathbb{R}, \ \alpha < \beta, \ \exists \, q \in \mathbb{Q} : \alpha < q < \beta$

Proof. $\alpha := (A, A')$, $\beta := (B, B')$, $A \subset B$, $A \subset \mathbb{Q}$, $B \subset \mathbb{Q}$,
$A \neq B \Rightarrow \exists \, q \in B$, $q \notin A$, $\alpha < q < \beta$.

Proposition. Any cut of $\mathbb{R}$ belongs $\mathbb{R}$.

Proof: Suppose that there exists some cut of $\mathbb{R}$: (S, T), where $s \in S \subset \mathbb{R}$ and $t \in T \subset \mathbb{R}$, $\forall (s,t)$: $s < t$ and $(S, T) \notin \mathbb{R}$. Define $A \subset S$, $A \subset \mathbb{Q}$, $B \subset T$, $B \subset \mathbb{Q}$, $A \cap B = \emptyset$, $A \cup B = \mathbb{Q}$. Then $(A, B) \neq (S, T) \Rightarrow \exists! \, p \in \mathbb{Q}$ $s < p < t$. This contraverse to dencely ordered of rationals.

The real numbers can be used as a measure of the length

1. Take an interval of length 1.
2. Construct intervals of rational length.
3. Others values of the length we obtain using the Dedekind cuts.

Result. For any interval we can be connected with real number as a length of the interval.

Corollary. The set of ordered real numbers coinsides to the stright line wiht given origin

Theorem (Cantor) The set of reals is uncountable

Proof. Suppose we can count the reals from $(0,1)$ and write the reals as a list of infinite decimal fractions:

$\alpha_1 : 0, a_{11}\, a_{12}\, a_{13} \ldots a_{1n} \ldots,$ $\alpha_2 : 0, a_{21}\, a_{22}\, a_{23} \ldots a_{2n} \ldots,$

$\ldots\ldots\ldots$

$\alpha_n : 0, a_{n1}\, a_{n2}\, a_{n3} \ldots a_{nn} \ldots$

$\ldots\ldots$

here a_{ij} is a digit from: $\{0,1,2,3,4,5,6,7,8,9\}$

Construct a number b:

$b : 0, b_1, b_2, b_3, \ldots b_n, \ldots$ where $b_1 \neq a_{11}$, $b_2 \neq a_{22}$, $b_3 \neq a_{33} \ldots$ $b_n \neq a_{nn}$.

So the list of α does not contain „b".

Summary

1. Natural numbers ($\mathbb{N}$)
2. Integer numbers ($\mathbb{Z}$)
3. Rational numbers ($\mathbb{Q}$)
   3.1 Countable
   3.2 Density
4. Dedekind cuts.
5. Real numbers ($\mathbb{R}$).
6. Cantor theorem.