Lecture 2

Properties of sequences

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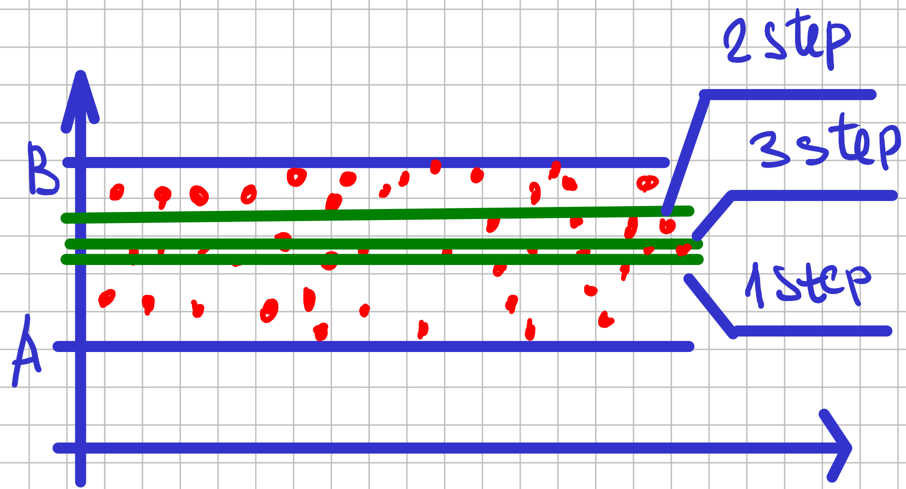
Theorem Bolzano - Weierstrass.

$\forall$ bounded $\{a_n\}_{n=1}^{\infty}$ contains a convergent subsequence.

Proof: $A = \inf\{a_n\}$, $B = \sup\{a_n\}$, consider $[A, \frac{A+B}{2}]$ and $[\frac{A+B}{2}, B]$. One of those interval contains a subsequence. Repetiting this process we obtain a convergent sequence of intervals which contains subsequence. Those subsequence is convergent due to convergency of the sequence of the intervals.



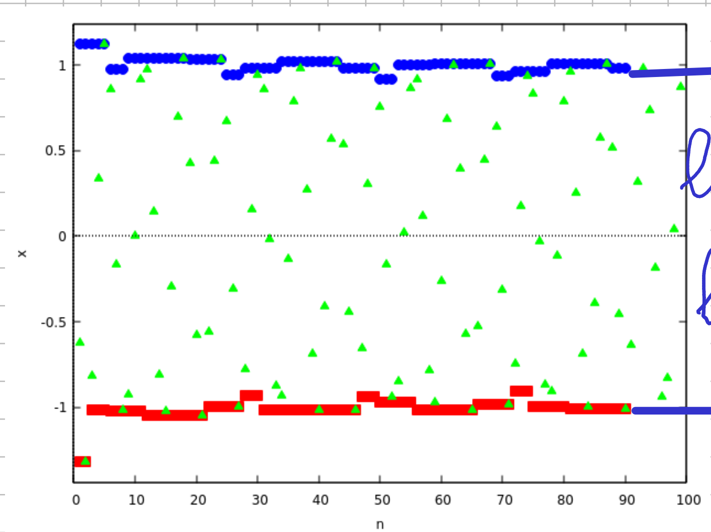
Def. A limit superior is a maximal of subsequental limits

Def. A limit inferior is a minimal of subsequental limits

__Example__ $a_n = \left(1 - \frac{1}{n}\right)\cos(n)$

$\overline{\lim_{n\to\infty}}\, a_n = 1$ or $\limsup_{n\to\infty} a_n = 1$

$\underline{\lim_{n\to\infty}}\, a_n = -1$ $\liminf_{n\to\infty} a_n = -1$.



$\limsup(a_n)$

$\liminf(a_n)$

Properties of limits

Here we give a sketch of proof several properties of limits:

if $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$ then;

1) $$\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = a+b$$

2) $$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = a\cdot b$$

Let's proof property 1):

From the definition of $\lim\limits_{n\to\infty} a_n$:

$$\forall \frac{\varepsilon}{2} > 0,\ \exists N_a,\ \forall n > N_a : |a_n - a| < \frac{\varepsilon}{2},$$

From the definition of $\lim\limits_{n\to\infty} b_n$:

$$\forall \frac{\varepsilon}{2} > 0,\ \exists N_b,\ \forall n > N_b : |b_n - b| < \frac{\varepsilon}{2}.$$

Take $N = \max\{N_a, N_b\} \Rightarrow$

$$\forall \varepsilon > 0\ \ \exists N : \forall n > N\ \ |a_n - a + b_n - b| < \varepsilon,$$

or:

$$|(a_n + b_n) - (a+b)| < \varepsilon \Rightarrow \lim\limits_{n\to\infty} (a_n + b_n) = a + b.$$

Let's proof 2):

From the definition of $\lim\limits_{n\to\infty} a_n$:

$$\forall \frac{\varepsilon}{2} > 0,\ \exists N_a,\ \forall n > N_a : |a_n - a| < \frac{\varepsilon}{2},$$

From the definition of $\lim\limits_{n\to\infty} b_n$:

$$\forall \frac{\varepsilon}{2} > 0,\ \exists N_b,\ \forall n > N_b : |b_n - b| < \frac{\varepsilon}{2}.$$

Take $N = \max\{N_a, N_b\}$ and consider:

$$|a_n b_n - ab| = |a_n b_n - a b_n + a b_n - ab| =$$

$$= |(a_n - a) b_n + a(b_n - b)| \leq |b_n| \cdot |a_n - a| + |a| \cdot |b_n - b| \leq$$

$$\leq \varepsilon \left(|b_n| + |a|\right).$$

Define $A = \sup_{n \in \mathbb{N}} |b_n| + |a|$, then

$|a_n b_n - ab| \leq A\varepsilon$, where $0 < A = \text{const}$.

Then $\forall \bar{\varepsilon} > 0$ (for example $\bar{\varepsilon} = A\varepsilon$) $\exists N;\ \forall n > N$

$|a_n b_n - ab| < \bar{\varepsilon}$.

$$\lim_{n \to \infty} a_n b_n = ab = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n.$$

as well as $\left\{\frac{a_n}{b_n}\right\} \equiv \left\{a_n \cdot \frac{1}{b_n}\right\}$ and

$$\{a_n - b_n\} \equiv \{a_n + (-b_n)\}$$

we proof that: $\lim\limits_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$,

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n.$$

Note: these properties are true for bounded sequences.

Counterexamples 1) $a_n = n-1$, $b_n = n$ and $\lim\limits_{n\to\infty}(a_n - b_n) = 1$.

2) $a_n = \frac{1}{n}$, $b_n = n\cos(n)$ $a_n b_n = \cos(n)$ and $\{a_n b_n\}$ is bounded.

Binomial theorem.

$$(x+y)^2=(x+y)(x+y)=x\cdot x+xy+yx+y^2=x^2+2xy+y^2$$

$$(x+y)^3=(x+y)(x+y)(x+y)=x\cdot x\cdot x+xxy+xyx+xyy+$$
$$+yxx+yxy+yyx+yyy=$$
$$=x^3+3x^2y+3xy^2+y^3.$$

$$(x+y)^n=x^n+a_{n-1}x^{n-1}y+a_{n-2}x^{n-2}y^2+\ldots+a_{n-k}x^{n-k}y^k+\ldots+y^n.$$

$a_{n-1}=n$. We can take multiplyer „y" from „n" different brackets $(x+y)$.

$a_{n-2}=\dfrac{n(n-1)}{2}$ We can take one „y" from „n" different brackets and second multiplyer „y" from different $(n-1)$ brackets.
But due to permutatrons of these two brakets we must divide $n(n-1)/2$!

$$a_k = \frac{n(n-1)(n-2)\dots(n-k+1)}{k!}$$

we can take y^k from $n(n-1)(n-2)\dots(n-k+1)$ brackets and due to ability of permutations these k brakets we must divide the number on $k!$.

So $a_k = \frac{n!}{(n-k)!\,k!} = \binom{n}{k}$, which is „n" choose „k"

It yields:

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \dots + \frac{n!}{(n-k)!\,k!}x^{n-k}y^k + \dots + y^n$$

Corollary:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \dots + \frac{n!}{(n-k)!\,k!}x^k + \dots + \frac{n!}{n!}x^n$$

2 „magic" limit.

Let's consider $x_n = \left(1 + \frac{1}{n}\right)^{n}$.

$n \to \infty$, $\frac{1}{n} \to 0$

Statement 1. This sequence increases monotonously as $n \to \infty$.

A sketch of a proof for this statement.

1) Expand the x_n using a binomial formula $(1+a)^n = 1 + na + \frac{n(n-1)}{2} a^2 + \dots$

2) Compare the expanded formulas for x_n and x_{n+1} term by term.

Proof. Rewrite as a sum:

$$x_n = 1 + n \cdot \frac{1}{n} + \frac{n\cdot(n-1)}{2!}\cdot\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\cdot\frac{1}{n^3} + \dots +$$

$$+ \frac{n!}{(n-k)!\,k!}\cdot\frac{1}{n^k} + \dots + \frac{n(n-1)\dots(n-n+1)}{n!\;n^n}$$

Rewrite every term as follows:

$$\frac{n\cdot(n-1)(n-2)\dots(n-k+1)}{k!}\,\frac{1}{n^k} = \frac{1}{k!}\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\dots\left(\frac{n-(k-1)}{n}\right) = \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{k-1}{n}\right)$$

Then:

$$x_n = 2 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \dots + \frac{1}{k!}\left(1-\frac{1}{n}\right)\dots\left(1-\frac{k-1}{n}\right) + \dots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\dots\left(1-\frac{n-1}{n}\right)$$

Compare term by term:

$$x_n = 2 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \dots + \frac{1}{k!}\left(1-\frac{1}{n}\right)\dots\left(1-\frac{k-1}{n}\right) + \dots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\dots\left(1-\frac{n-1}{n}\right)$$

$$x_{n+1} = 2 + \frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \dots + \frac{1}{k!}\left(1-\frac{1}{n+1}\right)\dots\left(1-\frac{k-1}{n+1}\right) + \dots + \frac{1}{n!}\left(1-\frac{1}{n+1}\right)\dots\left(1-\frac{n-1}{n+1}\right) + \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\dots\left(1-\frac{n}{n+1}\right).$$

Every term of x_{n+1} > connected term of x_n $\Longrightarrow$

$$x_{n+1} > x_n.$$

The statement is prooved!

Statement 2. The sequence $x_n = \left(1+\frac{1}{n}\right)^n$ is bounded.

The idea of a proof: compare formula for x_n as $n \to \infty$ and formulas for a sum of geometric series:

$$S = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} + \dots = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} + \dots\right) = \frac{1}{2} + \frac{1}{2} S,$$

$$\frac{1}{2} S = \frac{1}{2} \Rightarrow S = 1$$

Proof Let's compare x_n and y_n:

$$x_n = 2 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \dots + \frac{1}{k!}\left(1-\frac{1}{n}\right)\dots\left(1-\frac{k-1}{n}\right) + \frac{1}{n!}\left(1-\frac{1}{n}\right)\dots\left(1-\frac{n-1}{n}\right)$$

$$y_n = 2 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{k!} + \dots + \frac{1}{n!}$$

Obviously, $y_n > x_n$. Compare y_n and z_n:

$$y_n = 2 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{k!} + \dots + \frac{1}{n!}$$

$$z_n = 2 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^k} + \dots + \frac{1}{2^{n-1}}$$

$y_n < z_n$ and $z_n = 2 + \sum_{k=1}^{n-1}\left(\frac{1}{2}\right)^k < 2 + 1 = 3$

$$2 < \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n < 3.$$

Theorem by Jacob Bernoully (1683)

(Второй замечательный предел)

The sequence $x_n = \left(1+\frac{1}{n}\right)^n$ has a limit.

Proof Early we have shown that

1. x_n increases
2. $2 < x_n < 3$.

Therefore due to the theorem about convergency of bounded increased sequences

$$\exists \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \equiv e$$

$$e = 2{,}718281828459045\ldots$$

Summary

1. Teorem Bolzano-Weierstrass
2. lim sup, liminf.
3. Properties of limits.
4. $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.