Fixed point theorem

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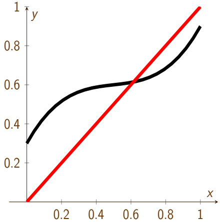
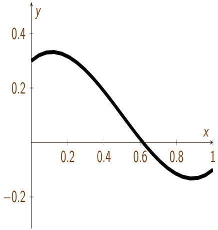
The fixed point theorem





Let $\phi(x)$
be a continuous map which acts from $[0, 1]$ into $[0, 1]$.
Then $\exists \xi \in [0, 1]$ such that:

$$\phi(\xi) = \xi.$$

Proof. Define

$$a = \min_{x \in [0,1]} (\phi(x)), \;\; a \geq 0;$$

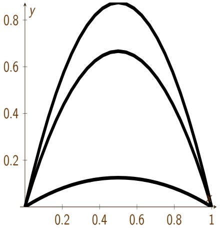
$$b = \max_{x \in [0,1]} (\phi(x)), \;\; b \leq 1;$$

$$f(x) \equiv \phi(x) - x,$$

$$f(0) \geq 0, \quad f(1) \leq 0, \Rightarrow$$

$$\exists \xi : f(\xi) = 0, \Rightarrow \phi(\xi) = \xi.$$

An example



Let's consider a map:

$$\phi(x) = 4\kappa(1-x)x, \quad \kappa > 0.$$

Find the maximum value of the map:

$$\phi'(x) = 4\kappa - 8\kappa x = 4\kappa(1-2x).$$

If the domain is $x \in [0,1]$, then the range of the map:

$$\phi\left(\frac{1}{2}\right) = 4\kappa\left(1-\frac{1}{2}\right)\frac{1}{2} = \kappa.$$

$$\phi : [0,1] \to [0,\kappa].$$

The example. Fixed points

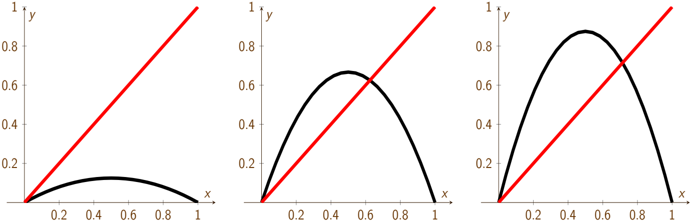


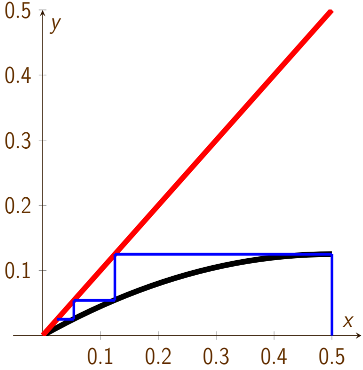
Figure: There are three typical picture for the solutions of the equation $x = \phi(x)$. The left graph connects to the case $0 < \kappa < 1/4$, the central graph connects to the case $1/4 < \kappa < 3/4$ and the right one for $3/4 < \kappa < 4$.

Fixed points

The ϕ contract the interval $[0, 1]$ into the interval $[0, \kappa]$.
It is easy to see that the map has two stationary points:

$$x = 4\kappa(1 - x)x, \Rightarrow x = 0,\ x = 1 - \frac{1}{4\kappa}.$$

Recurrent sequence



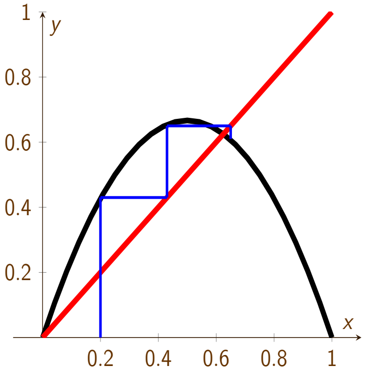
Let's consider the map for the given point x_0:

$$x_1 = 4\kappa(1 - x_0)x_0.$$

For recurrent case:

$$x_{n+1} = 4\kappa(1 - x_n)x_n;$$
$$x_0 = 0.5,\ x_1 = 0.125,$$
$$x_2 = 0.054,\ x_3 = 0.025.$$

Recurrent sequence



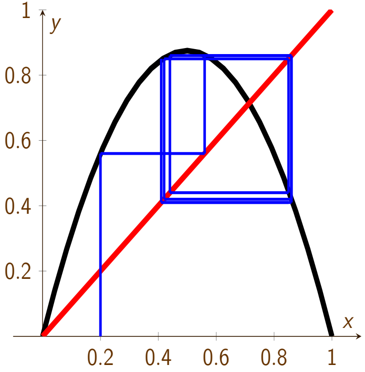
Let's consider the map for the given point x_0:

$$x_1 = 4\kappa(1 - x_0)x_0.$$

For recurrent case:

$$x_{n+1} = 4\kappa(1 - x_n)x_n;$$
$$x_0 = 0.2,\ x_1 = 0.34,$$
$$x_2 = 0.65,\ x_3 = 0.61.$$

Recurrent sequence



Let's consider the map for the given point x_0:

$$x_1 = 4\kappa(1 - x_0)x_0.$$

For recurrent case:

$$x_{n+1} = 4\kappa(1 - x_n)x_n;$$
$$x_0 = 0.2,\ x_1 = 0.56,$$
$$x_2 = 0.86,\ x_3 = 0.41,$$
$$x_4 = 0.85,\ x_5 = 0.45.$$

Contraction map

The derivatives at the fixed points.

$$
\begin{aligned}
\kappa &= 1/8, & x &= 0, & \phi'(0) &= 1/2; \\
\kappa &= 2/3, & x &= 0, & \phi'(0) &= 8/3; \\
& & x &= 5/8, & \phi'(5/8) &= -2/3; \\
\kappa &= 7/8, & x &= 0, & \phi'(0) &= 7/2; \\
& & x &= 5/7, & \phi'(5/7) &= -3/2.
\end{aligned}
$$

Contraction map on $\mathcal{X} \subset \mathbb{R}$.

A point $x \in \mathcal{X}$ is called **fixed point** of the map $\phi(x)$ if $\phi(x) = x$.
A map $\phi : \mathcal{X} \to \mathcal{X}$ is called **contraction map** if

$$|\phi(x_1) - \phi(x_2)| \leq k|x_1 - x_2|,\ 0 < k < 1.$$

Theorem

(Banach fixed point theorem) A contraction map $\phi : \mathcal{X} \to \mathcal{X}$ *has one and only one fixed point and* $\forall x_0 \in \mathcal{X}, \{x_n\}_{n=0}^{\infty} :$ $x_{n+1} = \phi(x_n),\ \lim_{n\to\infty} x_n = x_*,\ \phi(x_*) = x_*.$

Proof the theorem about fixed point

$$|\phi(x_{n+1}) - \phi(x_n)| \leq k|x_n - x_{n-1}| \leq \cdots \leq k^n|x_1 - x_0|,$$
$$|\phi(x_{n+1}) - \phi(x_n)| \to 0,\ n \to \infty.$$

Suppose $\exists y_* : \phi(y_*) = y_*$ and $x_* : \phi(x_*) = x_*$ $x_* \neq y_*$, then

$$|x_* - y_*| = |\phi(x_*) - \phi(y_*)| \leq k|x_* - y_*| \Rightarrow x_* \equiv y_*.$$

Newton's method

The main problem is to solve the nonlinear equation:

$$f(x) = 0$$

Suppose:

- $\exists x_* \in [a, b] : \; f(x_*) = 0;$
- $\forall x \in [a, b] : \; |f'(x)| > \text{const} > 0;$
- $\forall x \in [a, b] : |f(x)f''(x)| < (f'(x))^2.$

The Newton's method

Approximate the function by the linear one.

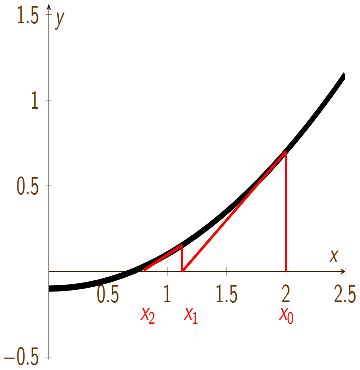
$$f'(x_0)(x - x_0) = y - f(x_0).$$

Now we obtain the linear equation for the variable $y = 0$:

$$f'(x_0)(x - x_0) + f(x_0) = 0, \quad x = -\frac{f(x_0)}{f'(x_0)} + x_0.$$

So we obtain the linear approximation for the solution.

A sequence of approaches



We can try
to use the approximation to next approach. In this case we can use the process:

$$x_{n+1} = -\frac{f(x_n)}{f'(x_n)} + x_n$$

The Newton method as the contracting map

Define

$$\phi(x) \equiv -\frac{f(x)}{f'(x)} + x.$$

This map is contracting if $|\phi'(x)| < 1$ or:

$$\left| -\frac{(f(x)')^2 - f(x)f''(x)}{(f'(x))^2} + 1 \right| < 1 \Rightarrow$$

$$\left| \frac{f(x)f''(x)}{(f'(x))^2} \right| < 1 \Rightarrow |f(x)f''(x)| < (f'(x))^2.$$

The convergence

An example:

$$f(x) \equiv x^2 - a, \quad a > 0,\ f'(x) = 2x.$$

Then

$$x_{n+1} = -\frac{x_n^2 - a}{2x_n} + x_n, \quad x_{n+1} = \frac{a}{2x_n} + \frac{x_n}{2}.$$

The convergence

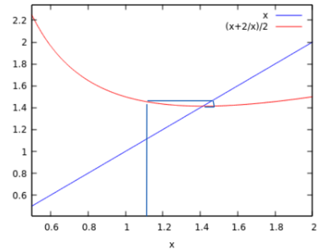
The convergence of this process:

$$x_{n+1} - x_n = \frac{a}{2x_n} + \frac{x_n}{2} - \frac{a}{2x_{n-1}} + \frac{x_{n-1}}{2},$$
$$\frac{1}{2}\left(1 - \frac{1}{x_n x_{n-1}}\right)(x_n - x_{n-1}).$$

$\sqrt{2}$

An approximation procedure.

$$x^2 = 2$$
$$x^2 + x^2 = 2 + x^2$$
$$2x^2 = x^2 + 2$$
$$x^2 = \frac{1}{2}(x^2+2)$$
$$x = \frac{1}{2x}(x^2+2)$$
$$x = \frac{1}{2}\left(x + \frac{2}{x}\right)$$



$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{2}{x_k}\right);$$

$x_1 = 1.1$, $x_2 = 1,459(09)$

$x_3 = 1,149037$

$x_4 = 1,14213\ldots$ $x_4^2 \approx 2,00000\ldots$

it's a Magic!

summary