Graphing

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On previous lecture

Monotonous functions

Extremum points and high-order derivatives

Geometric properties and derivatives of second-order

Taylor's formula

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o((x - x_0)^n).$$

Examples. $\sin(x)$

$$\sin'(x)|_{x=0} = \cos(x)|_{x=0} = 1$$
$$\cos'(x)|_{x=0} = -\sin(x)|_{x=0} = 0,$$
$$\sin(x) = \sin(x)|_{x=0} + \cos(x)|_{x=0}x + (-\sin(x)|_{x=0})\,\frac{x^2}{2!} -$$
$$(-\cos(x)|_{x=0})\,\frac{x^3}{3!} + (-\sin(x)|_{x=0})\,\frac{x^4}{4!} + (\cos(x)|_{x=0})\,\frac{x^5}{5!} + o(x^5),$$
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5);$$

Examples. $\log(1+x)$

$$\log'(1+x)|_{x=0} = \frac{1}{1+x}|_{x=0} = 1,$$

$$\left(\frac{1}{1+x}\right)'|_{x=0} = \frac{-1}{(1+x)^2}|_{x=0} = -1$$

$$\left(\frac{-1}{(1+x)^2}\right)'|_{x=0} = \frac{2}{(1+x)^3}|_{x=0} = 2,$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3).$$

Examples. $\sqrt{1+x}$

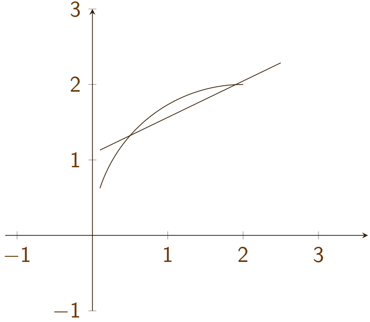
$$\left(\sqrt{1+x}\right)'|_{x=0} = \frac{1}{2\sqrt{1+x}}|_{x=0} = \frac{1}{2},$$

$$\left(\frac{1}{2\sqrt{1+x}}\right)'|_{x=0} = \frac{-1}{4(1+x)^{3/2}}|_{x=0} = \frac{-1}{4},$$

$$\left(\frac{-1}{4(1+x)^{3/2}}\right)'|_{x=0} = \frac{3}{8(1+x)^{5/2}}|_{x=0} = \frac{3}{8},$$

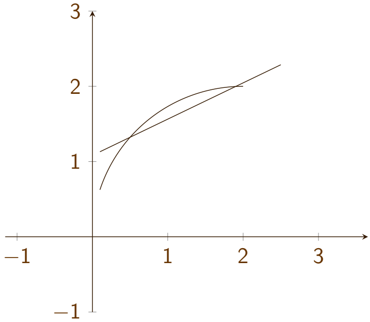
$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + o(x^3).$$

Monotonous function



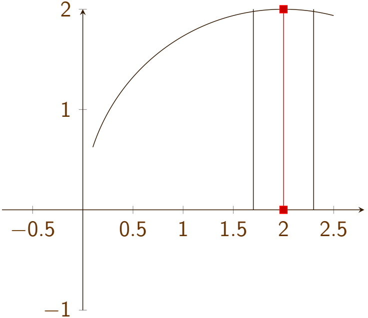
Theorem.
Let function $f(x)$ be defined and differential on (a, b). If $f'(x) > 0$ on (a, b) then the function increases on the interval, and if $f'(x) < 0$ on (a, b) then $f(x)$ decreases on (a, b).

A proof of the theorem



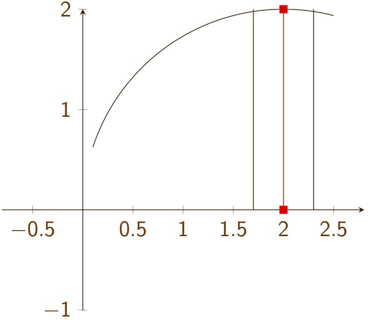
Proof.
Let $a < x_1 < x_2 < b$ then the Lagrange theorem climes $f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1)$ and $f'(\xi) > 0$. The opposite case can be considered by the same manner.

An extremum of a function



Let $f(x)$ be defined in a neighborhood of x_0. The x_0 is called a maximum (or minimum) of $f(x)$ if $\exists\,\delta > 0$:
$f(x_0 + \Delta) \leq f(x_0),\ \ |\Delta| < \delta$
(or $f(x_0 + \Delta) \geq f(x_0)$).
If the sign $\leq$ can be changed on $f(x) < f(x_0)$ then we will say a strong maximum and correspondingly strong minimum for the sign $f(x) > f(x_0)$

An extremum of a function



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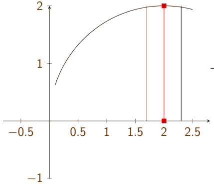
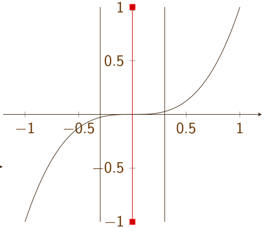
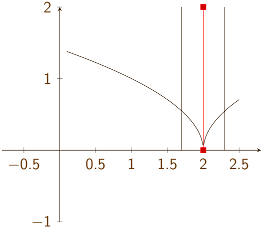
An extremum of a function

The point of maximum and minimum are called points of extremum.
The points of strong maximum and strong minimum are called strong extremum

Necessary conditions for the extremum



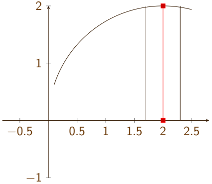


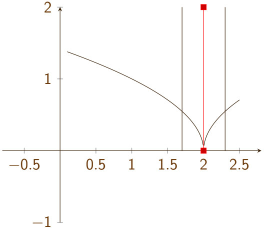


Theorem Let the point x_0 be an extremum point of $f(x)$, then the $f'(x_0) = 0$ or the derivative does not exits.

This theorem is a collocation of the Fermat's lemma.

Sufficient conditions for the strong extremum





Let $f(x)$ be continuous as $x \in (a, b)$ and differential in $(a, x_0) \cup (x_0, b)$. If $\exists \epsilon > 0, \forall \delta_1, \delta_2, \epsilon > \delta_1 > 0, \epsilon > \delta_2 > 0$: $\text{sign}(f'(x_0 - \delta_1)) \neq \text{sign}(f'(x_0 + \delta_2))$, then x_0 is a point of the strong extremum.

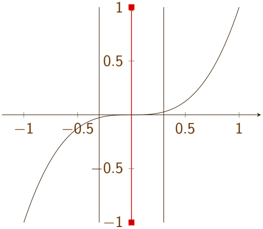
A proof

Let's consider $f'(x) > 0$ as $x \in (a, x_0)$ and $f'(x) < 0$ as $x \in (x_0, b)$. The Lagrange's theorem gives:

$$f(x) - f(x_0) = f'(\xi)(x - x_0).$$

If $x < x_0$, then $f'(\xi) > 0$, $x < \xi < x_0$, hence $f(x) - f(x_0) < 0$.
If $x > x_0$, then $f'(\xi) < 0$, $x_0 < \xi < x$, hence $f(x) - f(x_0) < 0$.
Therefore x_0 is a strong maximum.
The case of minimum can be considered by the same way.

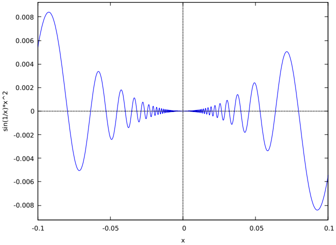
The points of increasing and decreasing



Definition

A point x_0 is called the point of increasing (or decreasing) of $f(x)$ if $\exists \delta > 0$ $f(x) - f(x_0) < 0, x_0 - \delta < x < x_0$ (or $f(x) - f(x_0) < 0$) and $f(x) - f(x_0) < 0, x_0 < x < x_0 + \delta$ (or $f(x) - f(x_0) > 0$).

The counterexample



Let's consider a function:

$$y(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0; \\ 0, & x = 0. \end{cases}$$

The point $x = 0$ is equal neither the point of increasing nor the point of decreasing.

The extremum points and high-order derivatives

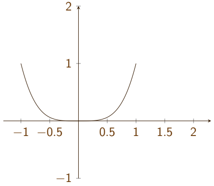
Theorem

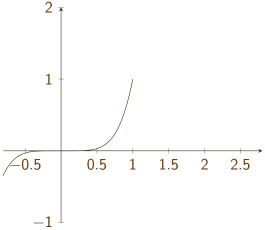
Let $f(x)$ be defined on neighborhood of x_0 and

$$f^{(k)}(x_0) = 0, \quad 0 < k < n-1, \quad f^{(n)}(x_0) \neq 0.$$

Then , if n is even, then x_0 is a strong maximum for $f^{(n)}(x_0) < 0$ and strong minimum for $f^{(n)}(x_0) > 0$. If n is odd then x_0 is a point of increasing if $f^{(n)}(x_0) > 0$ and a point of decreasing otherwise.

Proof of the theorem about extremum and high-order derivatives

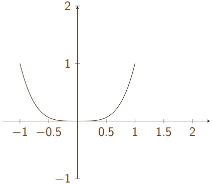


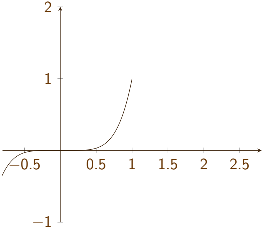


$$\Delta f = f(x_0 + \Delta) - f(x_0) = f^{(n)}(x_0)\frac{\Delta^n}{n!}$$

For even values of n the sign of the difference Δf is the same as the sign of the derivative. Hence x_0 maximum for negative $f^{(n)}(x_0)$ and minimum for positive one.

Proof of the theorem about extremum and high-order derivatives



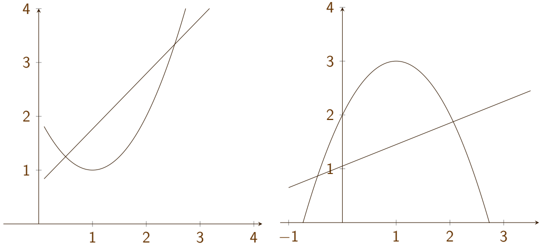


For odd values of n the signum of the Δf changes and therefore x_0 is an increasing point for positive $f^{(n)}(x_0)$ and a decresing point for negative one.

Corollaries

If $f'(x_0) > 0$ then x_0 is point of increasing.
If $f'(x_0) < 0$ then x_0 is point of decreasing.
If $f'(x_0) = 0$ and $f''(x_0) > 0$, then x_0 is point of minimum.
If $f'(x_0) = 0$ and $f''(x_0) < 0$, then x_0 is point of maximum.

A convexity and a concavity



Let's consider a secant line for a curve.

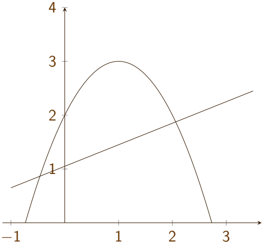
Proposition

If the segment of the curve lies under the secant line then a second derivative of the curve is positive.
If the segment of the curve lies upper than the secant line then a second derivative of the curve is negative.

A convexity and a concavity

The second derivative of the tangent line is equal to zero. Then the function with positive second derivative grows faster with respect to the tangent line. Then the tangent line lower than the curve at right side of the touch point and vice wise the negative second derivative means the function grows slowly with respect to tangent line and curve lower that the tangent line in the right side with respect to the touch point.

A convex curve



Let's consider a curve $f(x)$ on a segment $x \in [a, b]$.

Theorem about a convex curve

If the segment of the curve lies atop than any secant line then the curve is convex on this interval. If this curve has a second derivative, then the second derivative is negative.

A proof of the theorem about a convex curve

The equation for the secant line is follows:

$$y(x) = \frac{f(x_2)(x - x_1) + f(x_1)(x_2 - x)}{x_2 - x_1}$$

Consider a difference:

$$y(x) - f(x) = \frac{f(x_2)(x - x_1) + f(x_1)(x_2 - x) - f(x)(x_2 - x_1)}{x_2 - x_1} =$$

$$\frac{(f(x_2) - f(x))(x - x_1) - (f(x) - f(x_1))(x_2 - x)}{x_2 - x_1} =$$

$$\frac{f(\xi_2)(x_2 - x))(x - x_1) - f(\xi_1)(x - x_1)(x_2 - x)}{x_2 - x_1}$$

$$x_1 < \xi_1 < x < \xi_2 < x_2.$$

A proof of the theorem about a convex curve

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$$\frac{(f(x_2) - f(x))(x - x_1) - (f(x) - f(x_1))(x_2 - x)}{x_2 - x_1} =$$

$$\frac{f(\xi_2)(x_2 - x))(x - x_1) - f(\xi_1)(x - x_1)(x_2 - x)}{x_2 - x_1}$$

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$$\frac{f(\xi_2)(x_2 - x))(x - x_1) - f(\xi_1)(x - x_1)(x_2 - x)}{x_2 - x_1}$$

$$x_1 < \xi_1 < x < \xi_2 < x_2.$$

A proof of the theorem about a convex curve

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$$\frac{(f(x_2) - f(x))(x - x_1) - (f(x) - f(x_1))(x_2 - x)}{x_2 - x_1} =$$

$$\frac{f(\xi_2)(x_2 - x))(x - x_1) - f(\xi_1)(x - x_1)(x_2 - x)}{x_2 - x_1}$$

$$x_1 < \xi_1 < x < \xi_2 < x_2.$$

Summary

On previous lecture

Monotonous functions

Extremum points and high-order derivatives

Geometric properties and derivatives of second-order

We postpone the finish of the proof of the theorem about a convex curve up to the next lecture.