Derivatives 2.

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Implicit form of the function

The implicit form of the function $y(x)$ typically looks like

$$\Phi(x, y) = 0.$$

An example:

$$x^2 + y^2 - 1 = 0 \Rightarrow \begin{cases} y = \sqrt{1 - x^2}, & x \in [-1, 1); \\ y = -\sqrt{1 - x^2}, & x \in (-1, 1]. \end{cases}$$

Let's differentiate the formula over x:

$$2yy' + 2x = 0, \quad \Rightarrow y' = -\frac{x}{y}.$$

$$y^2 + x^2 = 1, \quad \Rightarrow (y')^2 = \frac{x^2}{y^2} = \frac{x^2}{1 - x^2},$$

then

$$(y')^2 = \frac{x^2}{1 - x^2}.$$

An example

$$x^3 + 4y^3 + 2x = 0,$$
$$3x^2 + 12y^2y' + 2 = 0,$$
$$y' = -\frac{2+3x^2}{12y^2}, \quad y = \sqrt[3]{-\frac{1}{2}x - \frac{1}{4}x^3},$$
$$y' = -\frac{2+3x^2}{12\left(\sqrt[3]{-\frac{1}{2}x - \frac{1}{4}x^3}\right)^2}.$$

General rules for differentiating of an implicit function

- Differentiate the formula $\Phi(x, y) = 0$ with respect to x (or y if it looks simple).
- Collect all terms which contains the derivative of function $\frac{dy}{dx}$ (or $\frac{dx}{dy}$).
- Try to use the obtained formula for defining the derivative.

A derivative of implicit function at a certain point

If one needs to find a value of a derivative at certain point (x_0, y_0). Then the recipe looks shorter.

- Differentiate the formula $\Phi(x, y) = 0$ with respect to x (or y if it looks simple). As a result one gets:

$$y' \frac{\partial \Phi(x, y)}{\partial y} + \frac{\partial \Phi(x, y)}{\partial x} = 0.$$

- Rewrite the formula in the following form:

$$y' = -\frac{\frac{\partial \Phi(x,y)}{\partial x}}{\frac{\partial \Phi(x,y)}{\partial y}}.$$

- Substitute the values of $x = x_0$ and $y = y_0$ into the right hand side of the formula.

An example

Let's find the derivative of the implicit function $y(x)$:

$$x^2 + y^2 - 1 = 0$$

at the point $(1/2, \sqrt{3}/2)$.
Differentiate the formula:

$$2x + 2yy' = 0, \quad y' = -\frac{x}{y}$$

Substitute the values of $x = 1/2$ and $y = \sqrt{3}/2$.

$$y' = -\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}.$$

One more example

Define the derivative y' of the implicit function $y(x)$ defined by:

$$x^3 + 4y^3 + 2x = 0$$

at the line $x = 1$.

- Take the formula for the derivative:

$$y' = -\frac{2 + 3x^2}{12y^2}$$

- Find the intersection of the line $x = 1$ and the algebraic curve:

$$1 + 4y^3 + 2 = 0, \quad y = -\sqrt[3]{\frac{3}{4}}.$$

One more example

- Substitute the values $x = 1$ and $y = -\sqrt[3]{\frac{3}{4}}$ into the formula for the derivative:

$$y' = -\frac{5}{12\left(\sqrt[3]{\frac{3}{4}}\right)^2} = -\frac{5}{3\sqrt[3]{36}}.$$

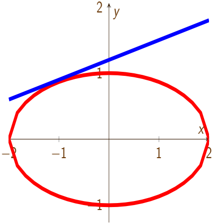
Parametric form of the function

General parametric form for a curve $y(x)$:

$$x = x(t), \quad y = y(t).$$

In geometry and physics the parametric form of the function is used elsewhere.

Parametric form of the ellipse



$$x = a\cos(t),\ \ y = b\cos(t), \qquad \frac{1}{a^2}x^2 + \frac{1}{b^2}y^2 = 1.$$

To define a tangent
line one should find the derivative $\frac{dy}{dx}$.

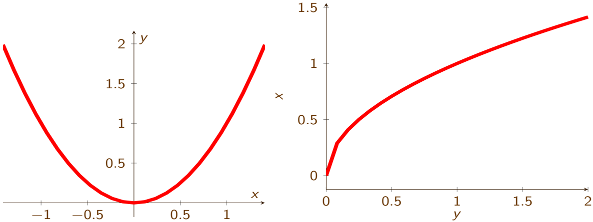
Differentiating of the function in the parametric form

$$dx = x'(t)dt, \quad dy = y'(t)dt, \quad \frac{dy}{dx} = \frac{y'(t)}{x'(t)}.$$

An example

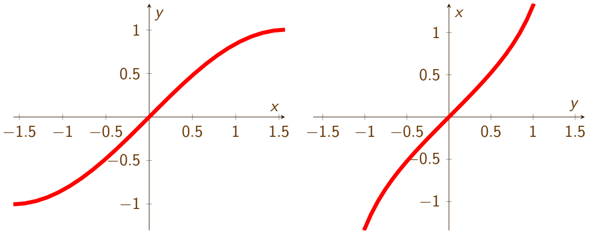
$$x = a\cos(t),\ y = b\sin(t),$$
$$dx = -a\sin(t)dt,\ dy = b\cos(t)dt,$$
$$\frac{dy}{dx} = -\frac{b\cos(t)}{a\sin(t)}, \quad \frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y}.$$

An inverse function. Examples



- $y = x^2$, an implied domain $x \in (-\infty, \infty)$ and range is $y \in [0, \infty)$.
- The inverse function is $x = \sqrt{y}$, an implied domain $y \in [0, \infty)$ and range is $x \in [0, \infty)$.

An inverse function. Examples



- $y = \sin(x)$, $x = \arcsin(y)$, $x \in [-\pi/2, \pi/2]$, $y \in [-1, 1]$.

An inverse function.

Definition

Let $y(x)$ be a continuous monotonous increased (or decreased) function. We call $x(y)$ the inverse function of $y(x)$ if $y(x(y)) \equiv y$ where domain of $y(x)$ is $x \in [a, b]$, range of y is $y \in [y(a), y(b)]$ ($y \in [y(b), y(a)]$).

Derivatives of inverse function

Let's assume a function $y(x)$ has an inverse function

$$y(x(y)) = y, \quad \frac{dy}{dx} \equiv \frac{dx}{dy} y'(x) = 1 \quad \frac{dx}{dy} = \frac{1}{y'}.$$

Examples.

$$\frac{d}{dx} \arcsin(sin(x)) = 1, \quad \arcsin'(\sin(x)) \cos(x) = 1,$$

$$\arcsin'(\sin(x)) = \frac{1}{\cos(x)}, \; \cos(x) = \sqrt{1 - \sin^2(x)},$$

Denote $\sin(x) = y$, then

$$\arcsin'(y) = \frac{1}{\sqrt{1 - y^2}}.$$

An example

$$\arctan(\tan(x))) = x,$$

$$\arctan'(\tan(x)))\frac{1}{\cos^2(x)} = 1, \quad \arctan'(\tan(x))) = \cos^2(x),$$

$$\frac{1}{\cos^2(x)} - 1 = \frac{\sin^2(x)}{\cos^2(x)} = \tan^2(x), \Rightarrow 1 + \tan^2(x) = \frac{1}{\cos^2(x)},$$

$$\cos^2(x) = \frac{1}{1 + \tan^2(x)}, \quad y = \tan(x), \Rightarrow \arctan'(y) = \frac{1}{1 + y^2}.$$

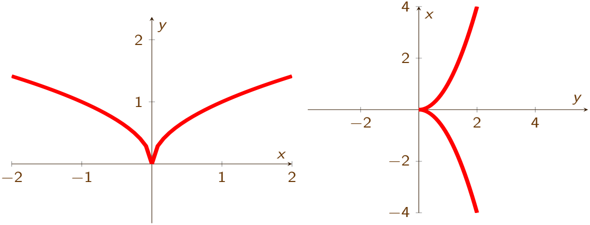
Theorem about an inverse function

Theorem

Let function $y = f(x)$ has a derivative in a point x_0 and $f(x)$ is monotonous in a neighborhood of x_0. Then an inverse function $f^{-1}(y)$ exists in a neighborhood of $y_0 = f(x_0)$. The derivative of the inverse function:

$$\left(f^{-1}(y)\right)' = \frac{1}{f'(x)}.$$

Counterexamples



- If $f(x)$ is not monotonous on the interval, then one cannot construct one-valued inverse function.
- if $f(x)$ has not a derivative at x_0, then does not exist an one-valued inverse function.

A proof of the theorem about the inverse function

Define $(x_0 + h, x_0 - h)$ an interval of monotonous of the function $f(x)$. Assume for a distinctness that the function monotonously increases.
The function is continuous on the interval $(x_0 + h, x_0 - h)$ then from the theorem about an intermediate value:

$$\forall y \in (f(x_0 - h), f(x_0 + h)), \exists x \in (x_0 - h, x_0 + h) : f(x) = y.$$

Assume that $\exists y_*$: $f(x_*) = y_*$ and $f(x^*) = y_*$, and $x_* < x^*$. Then due to the monotonous growth: $f(x_*) < f(x*)$. We obtain a contradiction.

A proof of existence of derivative for an inverse function

$$\Delta x = x - x_0,$$

$$\lim_{\Delta x\to 0}\frac{\Delta x}{f(x_0+\Delta x)-f(x_0)} = \lim_{\Delta x\to 0}\frac{\Delta x}{f'(x_0)\Delta x + o(\Delta x)} =$$

$$\frac{1}{\lim_{\Delta x\to 0}\frac{f'(x_0)\Delta x+o(\Delta x)}{\Delta x}} = \frac{1}{f'(x_0)}.$$

Derivatives of second order

The second order derivative is denoted as follows:

$$f''(x) \equiv \frac{d^2 f}{dx^2}.$$

A question. Why the nominator and denominator are written in the different manner in the last formula?

$$f''(x) \equiv \lim_{\Delta \to 0} \frac{f'(x+\Delta) - f'(x)}{\Delta} =$$

$$\lim_{\Delta \to 0} \left(\frac{f(x+2\Delta) - f(x+\Delta)}{\Delta} - \frac{f(x+\Delta) - f(x)}{\Delta} \right) \frac{1}{\Delta}$$

$$\lim_{\Delta \to 0} \frac{f(x+2\Delta) - 2f(x+\Delta) + f(x)}{\Delta^2} \equiv \frac{d^2 f}{dx^2}.$$

Derivatives of the second order

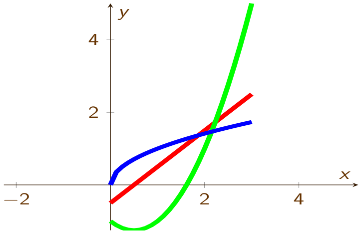
Define a difference operator which acts on the functions:

$$
\begin{aligned}
D(f(x)) &:= f(x+\Delta) - f(x). \\
D^2(f) &\equiv D(D(f(x))) = D(f(x+\Delta) - f(x)) \\
&= D(f(x+\Delta)) - D(f(x)) = \\
&\quad (f(x+2\Delta) - f(x+\Delta)) - (f(x+\Delta) - f(x)).
\end{aligned}
$$

So, the formula for the second derivative can be written as follows:

$$\frac{d^2 f}{dx^2} = \lim_{\Delta \to 0} \frac{D^2(f(x))}{\Delta^2}.$$

A geometrical sense of the second-order derivative



Let's consider a linear function:

$$y(x) = kx + b,\ \ y'(x) = k,\ \ y''(x) \equiv 0.$$

The same for a quadratic function:

$$y(x) = x^2 + bx + c, \quad y'(x) = 2x + b, \quad y''(x) \equiv 2.$$

The same for a square root:

$$y(x) = \sqrt{x}, \quad y'(x) = \frac{1}{2\sqrt{x}}, \quad y'' = -\frac{1}{4x\sqrt{x}}.$$

Observations: If the function increases faster than straight line then second derivative is positive.

A physical sense of the second derivative

Consider a straight line motion:

- Let $x(t)$ be a dependency of distance on time.
- $\dot{x}$ is a velocity.
- $\ddot{x}$ is an acceleration.

An example. A vertical motion

- $\ddot{x} = -g$ is an acceleration.
- $\dot{x} = -gt$ is a velocity.
- $x = -g\frac{t^2}{2}$ is an instant coordinate.

Derivatives of high-order. An example

$$
\begin{aligned}
F(x) &= x^3 + 2x^2 + \sin(x), \\
F'(x) &= 3x^2 + 4x + \cos(x), \\
F''(x) &= 6x + 4 - \sin(x), \\
F'''(x) &= 6 - \cos(x), \\
F^{(4)}(x) &= \sin(x).
\end{aligned}
$$

High-order derivatives for parametric given functions

$$
\begin{aligned}
y &= y(t),\ x = x(t), \Rightarrow y(x) = y(t(x)), \\
\frac{dy}{dx} &= \frac{dt}{dx}\frac{d}{dt}(y(t)) = \frac{1}{\frac{dx}{dt}}\frac{d}{dt}(y(t)) = \frac{y'(t)}{x'(t)}, \\
\frac{d^2y}{dx^2} &= \frac{dt}{dx}\frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right) = \frac{1}{x'(t)}\frac{y''(y)x'(t) - y'(t)x''(t))}{(x'(t))^2} = \\
& \frac{y''(y)x'(t) - y'(t)x''(t))}{(x'(t))^3}.
\end{aligned}
$$

A formula for differentiation

$$
\begin{aligned}
y &= y(t),\ x = x(t), \\
\frac{d^n y}{dx^n} &= \frac{dt}{dx}\frac{d}{dt}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) \equiv \frac{1}{\frac{dx}{dt}}\frac{d}{dt}\left(\frac{d^{n-1}y}{dx^{n-1}}\right).
\end{aligned}
$$