Improper integrals-2

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Improper integrals

Improper integrals on finite interval for the function $f(x)$ with no limit at $x = b$:

$$\int_a^b f(x)dx = \lim_{t \to b-0} \int_a^t f(x)dx.$$

Typical integrands are following.

- The integrand has the vertical asymptote at $x = b$: $f(x) = \frac{1}{\sqrt{b-x}}$
- The integral has no limit at the point $x = b$: $f(x) = \cos\left(\frac{1}{b-x}\right)$.

Improper integrals

Improper integrals on infinite interval for the function $f(x)$:

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx.$$

Typical integrands are following.

- The integrand tends to 0 as $x \to \infty$: $f(x) = \frac{1}{x^\alpha}$, $\alpha > 1$.
- The integrand oscillates fast as $x \to \infty$: $f(x) = \cos(x^\beta)$, $\beta > 1$.

Techniques for the improper integrals

- Changing variables:

$$\int_e^\infty \frac{dx}{x\log^2(x)} = \left| y = \log(x), \quad dy = \frac{dx}{x} \right| = \int_1^\infty \frac{dy}{y^2} = -\frac{1}{y}\Big|_{y=1}^{y=\infty} = 1.$$

- A comparison with known integrals.

$$\int_1^\infty e^{-x^2/2}dx < \int_1^\infty e^{-x/2}dx = -2e^{-x/2}\Big|_{x=1}^{x=\infty} = \frac{2}{\sqrt{e}}.$$

Corollary: the integral $\int_{-\infty}^{\infty} e^{-x^2/2}dx$ exists.

Techniques for the improper integrals

The usage the integration by parts:

$$\int_x^\infty e^{-t^2/2}dt = \int_x^\infty e^{-t^2/2}\frac{tdt}{t} = \int_x^\infty e^{-t^2/2}\frac{d\left(\frac{t^2}{2}\right)}{t} =$$

$$-\int_x^\infty \frac{d(e^{-t^2/2})}{t} = -\frac{e^{-t^2/2}}{t}\Bigg|_{t=x}^{t=\infty} - \int_x^\infty \frac{e^{-t^2/2}}{t^2}dt =$$

$$\frac{e^{-t^2/2}}{t} + \int_x^\infty \frac{e^{-t^2/2}}{t^2}dt = \frac{e^{-x^2/2}}{x} - \frac{e^{-x^2/2}}{x^2}\int_x^\infty \frac{e^{x^2/2-t^2/2}}{\frac{t^2}{x^2}}dt$$

Techniques for the improper integrals

$$\int_x^\infty \frac{e^{x^2/2-t^2/2}}{\frac{t^2}{x^2}}dt = \left|1 \le \frac{t^2}{x^2}\right| < \int_x^\infty e^{-(t+x)(t-x)/2}dt <$$

$$\int_x^\infty e^{-2x(t-x)/2}dt = |t-x=y| =$$

$$\int_0^\infty e^{-x\,y}dy = \left.\frac{e^{-x\,y}}{-x}\right|_{y=0}^{y=\infty} = \frac{1}{x}.$$

It yields:

$$\int_x^\infty e^{-t^2/2}dt \sim \frac{e^{-x^2/2}}{x},\; x \to \infty.$$

The Cauchy test

If $\forall\epsilon > 0\ \exists b(\epsilon) \geq a$ and $\forall b_2 > b_1 > b$:

$$\left|\int_{b_1}^{b_2} f(x)dx\right| < \epsilon,$$

then the improper integral

$$\int_a^{\infty} f(x)dx$$

converges. **Proof** of this theorem is based on the definition of the limit

$$\int_a^{\infty} f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx.$$

The Cauchy test

This means $\forall\epsilon > 0\ \exists N(\epsilon) > 0$:

$$\lim_{t\to\infty}\int_a^t f(x)dx = \int_a^b f(x)dx + \varepsilon,\ \ b(\varepsilon).$$

then If $\forall\epsilon > 0\ \exists b(\epsilon) \geq a$ and $\forall b_2 > b_1 > b$:

$$\left|\int_{b_1}^{b_2} f(x)dx\right| < \epsilon,$$

and contra versa if $\forall\epsilon > 0\ \exists b(\epsilon) \geq a$ and $\forall b_2 > b_1 > b$:

$$\left|\int_{b_1}^{b_2} f(x)dx\right| < \epsilon,$$

then

$$\lim_{t\to\infty}\int_a^t f(x)dx = \int_a^b f(x)dx + \varepsilon,\ \ b > N(\varepsilon)$$

The Cauchy test for bounded intervals

If $\forall\epsilon > 0\ \exists\delta(\epsilon)$ and $\forall b - \delta < b_1 < b_2 < b$:

$$\left| \int_{b_1}^{b_2} f(x)dx \right| < \epsilon,$$

then the improper integral

$$\int_a^b f(x)dx$$

converges.

Proof of this theorem is based on the definition of the limit

$$\int_a^b f(x)dx = \lim_{t\to b} \int_a^t f(x)dx.$$

The Cauchy test for bounded intervals

This means $\forall \epsilon > 0 \; \exists \delta(\epsilon) > 0$:

$$\lim_{t \to b} \int_a^t f(x)dx = \int_a^{b-\delta} f(x)dx + \epsilon.$$

then If $\forall \epsilon > 0 \; \exists \delta(\epsilon) \geq a$ and $\forall b - \delta < b_1 < b_2 < b$:

$$\left| \int_{b_1}^{b_2} f(x)dx \right| < \epsilon,$$

and contra versa if $\forall \epsilon > 0 \; \exists b(\epsilon) \geq a$ and $\forall b_2 > b_1 > b$:

$$\left| \int_{b_1}^{b_2} f(x)dx \right| < \epsilon,$$

then

$$\lim_{t \to b} \int_a^t f(x)dx = \int_a^{b-\delta} f(x)dx + \epsilon.$$

Dirichlet's test

Theorem

Let

- $f(x) \in \mathbf{C}[a, \infty)$
- $\exists\, F(x) : \ F'(x) = f(x) : \ |F(x)| < const, x \geq a;$
- $g(x) \in \mathbf{C}^1[a, \infty);$
- $g(x_1) > g(x_2) \forall a < x_1 < x_2;$
- $\lim_{x\to\infty} g(x) = 0.$

Then the integral

$$\int_a^\infty f(x)g(x)dx$$

is convergent.

A proof of the Dirichlet's test

$$\int_a^\infty f(x)g(x)dx = F(x)g(x)|_{x=a}^{x=\infty} - \int_a^\infty F(x)g'(x)dx =$$

$$-F(a)g(a) - \int_a^\infty F(x)g'(x)dx,$$

where

$$\int_a^\infty |F(x)g'(x)|dx \leq \sup_{x\in[a,\infty)} (|F(x)|) \int_a^\infty |g'(x)|dx$$

$$= - \sup_{x\in[a,\infty)} (|F(x)|) \int_a^\infty g'(x)dx = \sup_{x\in[a,\infty)} (|F(x)|)g(a).$$

Gamma-function

Let's consider:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \ \ z > 0.$$

$$\Gamma(z+1) = \int_0^\infty t^z \, e^{-t} dt = -t^z \, e^{-t}\Big|_{t=0}^{t=\infty} + z \int_0^\infty t^{z-1} e^{-t} dt.$$

It yields:

$$\Gamma(z+1) = z\Gamma(z).$$

$$\Gamma(1) = \int_0^\infty e^{-t} dt = 1.$$

Hence:

$$\Gamma(n+1) = n!, \ \ n \in \mathbb{N}.$$

Gamma-function

Theorem

The gamma-function is defined for all $z > 0$.

Proof. If $(n-1) < z \leq n$, $n \in \mathbb{N}$ then:

$$\Gamma(z) = (z-1)(z-2) \times \cdots \times (z-(n+1)) \int_0^\infty t^\alpha e^{-t} dt,$$

$-1 < \alpha \leq 0$. Let's consider the sum of the integrals:

$$\int_0^\infty t^\alpha e^{-t} dt = \int_0^1 t^\alpha e^{-t} dt + \int_1^\infty t^\alpha e^{-t} dt.$$

Gamma-function

$$\int_0^1 t^\alpha e^{-t} dt < e \int_0^1 t^\alpha dt = e \frac{t^{\alpha+1}}{\alpha+1}\Big|_0^1 = e.$$

Let's estimate the following integral using the Dirichlet's test as $-1 < \alpha < 0$:

$$\int_1^\infty t^\alpha e^{-t} dt.$$

Here $g(t) = t^\alpha \in \mathbf{C}^1[1, \infty)]$, t^α decreases monotonously and $\lim_{t\to\infty} t^\alpha = 0$. The $F(t) = e^{-t}$ has a bounded antiderivative. According to the Dirichlet's test one can claim that the integral exists.

Theorem is proved.

Mouivre-Stirling approximation

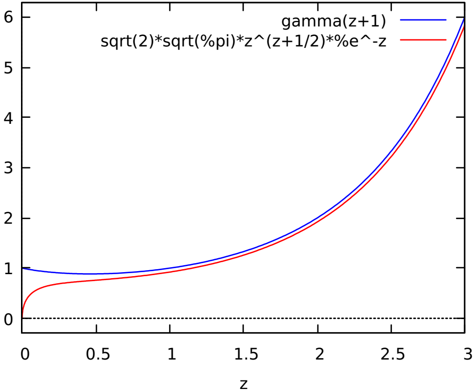

Figure: The gamma-function ang Mouivre-Stirling approximation:
$\Gamma(z+1) \sim \sqrt{2\pi z}\left(\frac{z}{e}\right)^{z}, \quad z \to \infty.$

Summary

On the previous lecture

Technique for improper integrals

Dirichlet's test

Gamma-function