Differentiating functions in a parametric form.

$$\left.\begin{array}{ll} y = y(t), & dy = y'(t)\,dt \\ x = x(t) & dx = x'(t)\,dt \end{array}\right\} \Rightarrow \frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$

Example: $$\left.\begin{array}{ll} y = \sin(t) & dy = \cos(t)\cdot dt \\ x = \cos(t) & dx = -\sin(t)\,dt \end{array}\right) \Rightarrow \frac{dy}{dx} = -\frac{\cos(t)}{\sin(t)}$$

or: $$\frac{dy}{dx} = -\frac{x}{y}. \quad , \quad \frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}} \quad ,$$

$$\frac{dy}{dx} = \frac{x}{\sqrt{1-x^2}} \quad ,$$

$$(y')^2 = \frac{x^2}{1-x^2}.$$

Guideline for differetiating of function in a parametric form.

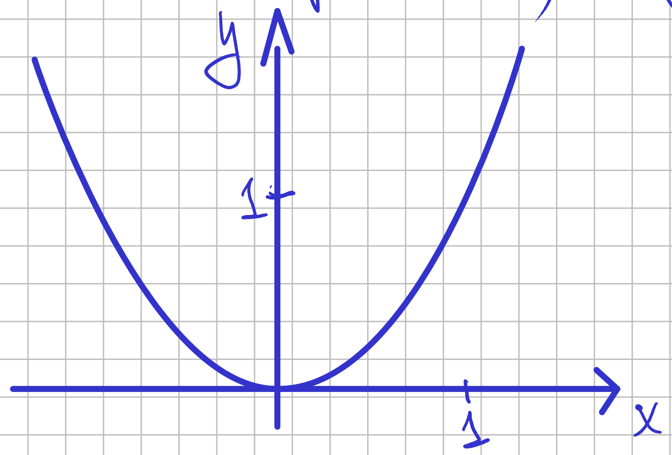
$y = y(t), \quad x = x(t) \Rightarrow dy = y'(t)dt,$

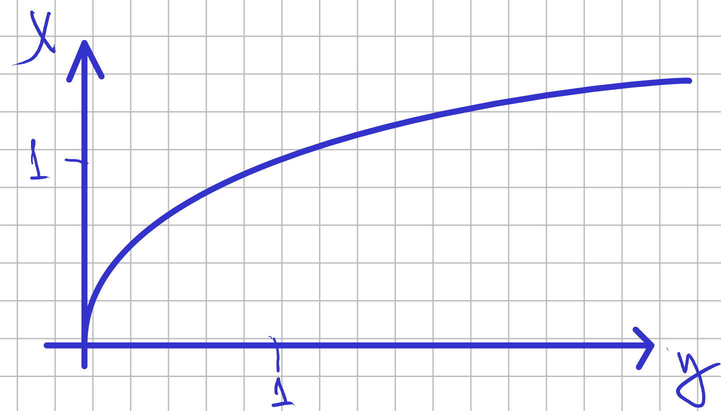
$dx = x'(t)dt \Rightarrow \frac{dy}{dx} = \frac{y'(t)}{x'(t)} \equiv \varphi(t)$

as a result one gets a parametric formula for the derivative.

An inverse function.

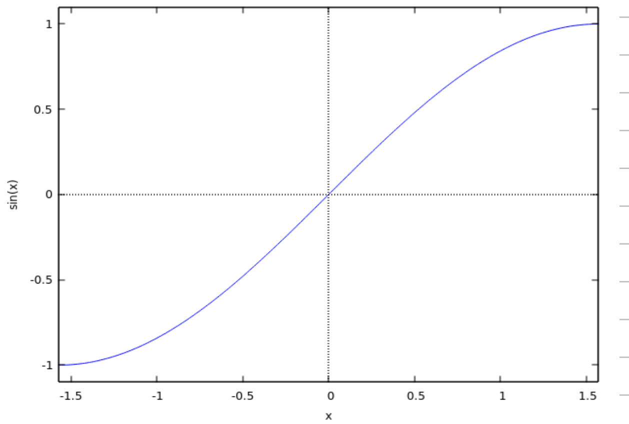
Example. 1) $y = x^2 \Rightarrow x = \sqrt{y}, \quad y > 0.$

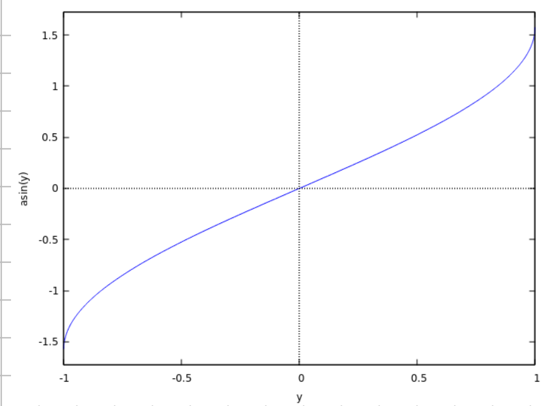




2) $y = \sin(x) \Rightarrow x = \arcsin(y) \qquad x \in [-\frac{\pi}{2}, \frac{\pi}{2})$

$y \in (-1, 1)$





We call $x(y)$ the inverse function of $y(x)$ if

$$y(x(y)) \equiv y$$

Example:

1) $y = x^2$, $x = \sqrt{y}$ $y(x(y)) = y$ $y = x^2 = (\sqrt{y})^2 \Rightarrow y \equiv y$.

2) $y = \sin(x)$, implied domain: $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$,

range of function: $y \in [-1, 1]$

inverse function $x = \arcsin(y)$, implied domain $y \in [-1, 1]$,

range of the function $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Assume $y = f(x)$ has an inverse function $f^{-1}(y)$

then $f^{-1}(f(x)) = x$,

$$\frac{d}{dx}\left(f^{-1}(f(x))\right) = 1 \Rightarrow f^{-1}(f(x)) \cdot f'(x) = 1 .$$

Denote $f^{-1}(y) = h(y)$, then :

$h(f(x)) = x$ and $\frac{d}{dx} h(f(x)) = 1$,

$$h'(f(x)) \cdot f'(x) = 1 \Rightarrow h'(f(x)) = \frac{1}{f'(x)}$$

Examples:

1) $\frac{d}{dx}\arcsin(\sin(x)) = 1 \Rightarrow \arcsin'(\sin(x))\cdot\cos(x) = 1,$

$\arcsin'(\sin(x)) = \frac{1}{\cos(x)},\qquad \cos(x) = \sqrt{1-\sin^2(x)}$

denote $\sin(x) = y \Rightarrow \arcsin'(y) = \frac{1}{\sqrt{1-y^2}}.$

2) $\operatorname{arctg}(\operatorname{tg}(x)) = x$

$$\operatorname{arctg}'(\operatorname{tg}(x))\,\frac{1}{\cos^2 x} = 1 \Rightarrow \operatorname{arctg}'(\operatorname{tg}(x)) = \cos^2(x)$$

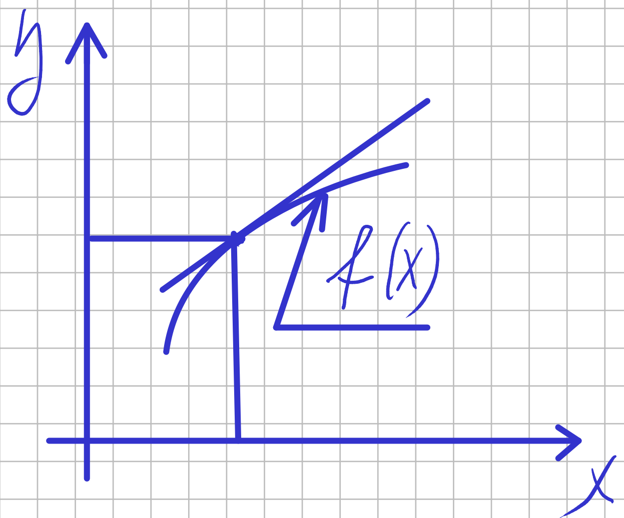
$$\frac{1}{\cos^2 x} - 1 = \frac{\sin^2(x)}{\cos^2(x)} = \operatorname{tg}^2 x \Rightarrow \quad 1 + \operatorname{tg}^2(x) = \frac{1}{\cos^2(x)}$$

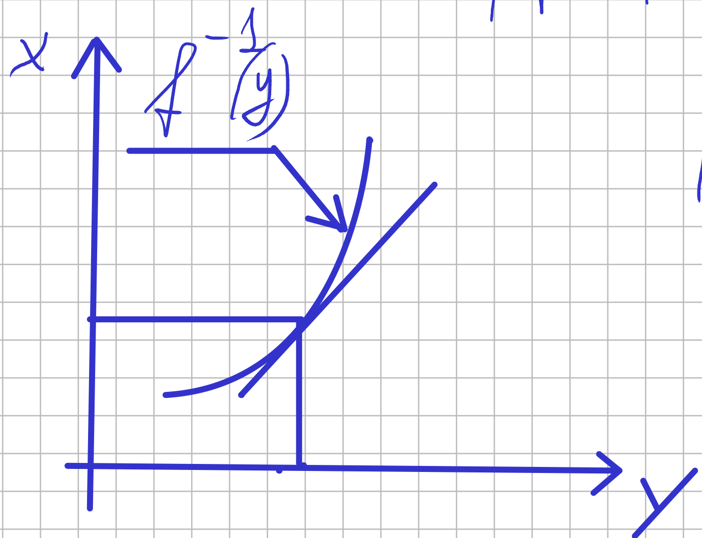
$$\cos^2(x) = \frac{1}{1+\operatorname{tg}^2 x}, \quad y = \operatorname{tg}^2 x \Rightarrow \operatorname{arctg}'(y) = \frac{1}{1+y^2}.$$

Theorem Let function $f(x)$ has a derivative at point x_0 and $f(x)$ is monotonous in a neighbourhood of x_0. Then an inverse function $f^{-1}(x)$ exists in a neighbourhood of x_0. The derivative of the inverse function equals

$$\left(f^{-1}(x)\right)' = \frac{1}{f'(x)}.$$

$$|f'(x)| < 1 \Rightarrow \left|\left(f^{-1}(y)\right)'\right| > 1$$

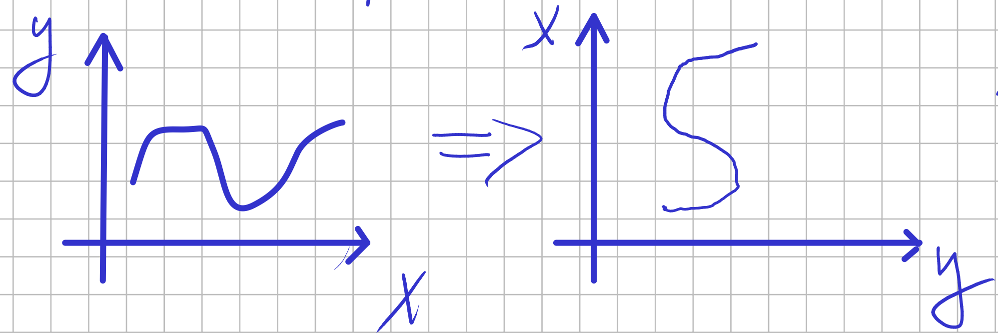




$$f'(x) = 0$$

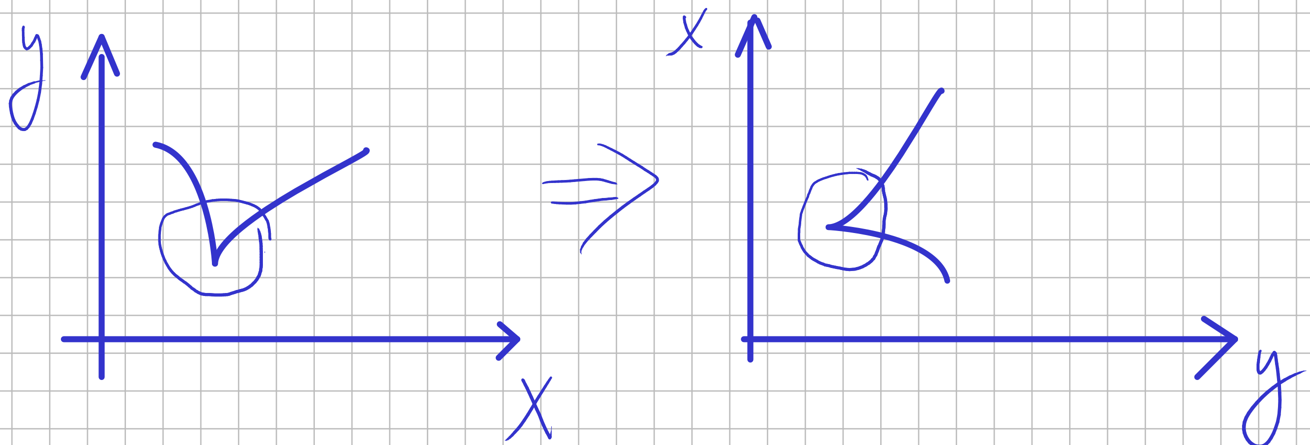
$$\left(f^{-1}(x)\right)' = \infty$$

Counterexamples 1) if $f(x)$ is not monotonous on an interval then one cannot construct one-valued inverse function.

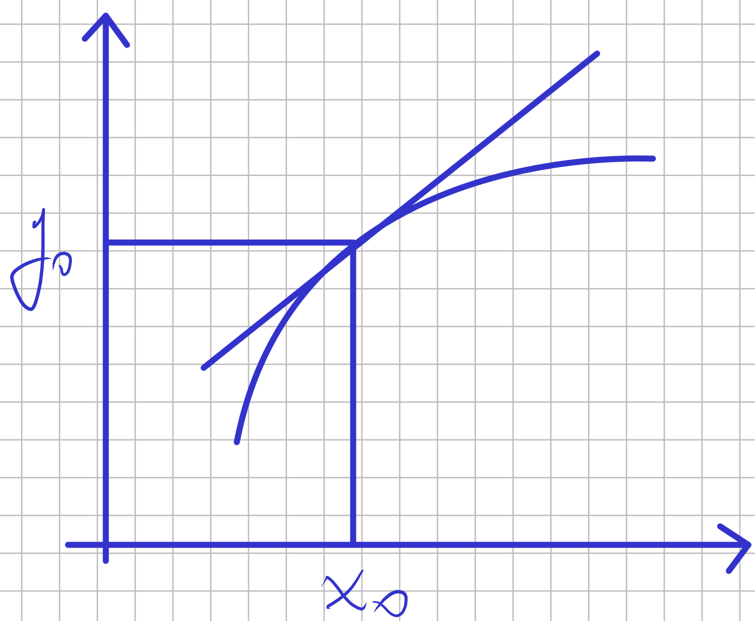


$x(y)$ is multivalued

2) if $f(x)$ has not a derivative at x_0:



Tangent line for a curve



$$k = f'(x_0)$$

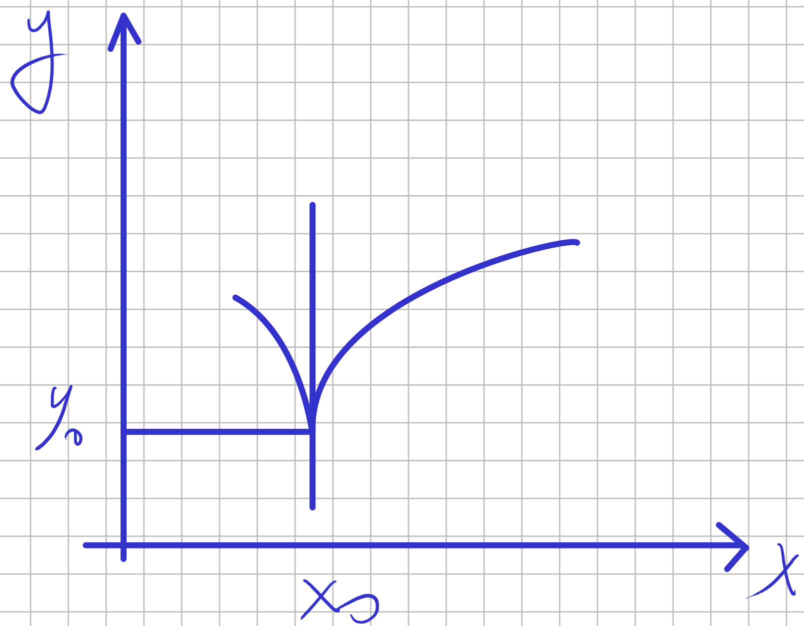
$$y - y_0 = f'(x_0) \cdot (x - x_0)$$

$$f(x) = x^2, \quad x_0 = 2$$

$$k = 2x\Big|_{x=2} = 4$$

$$y = 4,$$

$$y - 4 = 4\,(x - 2)$$



if $f'(x_0) = \infty \Rightarrow$ the tangent line is vertical.

Derivatives of high order.

$$f''(x) = \left(f'(x)\right)' \Rightarrow f^{(n)}(x) = \left(f^{(n-1)}(x)\right)'$$

$$\lim_{\Delta x \to 0} \frac{\dfrac{f(x+\Delta x) - f(x)}{\Delta x} - \dfrac{f(x) - f(x-\Delta x)}{\Delta x}}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - 2f(x) - f(x-\Delta x)}{\Delta x^2} \equiv \frac{d^2 f}{dx^2}$$

$$f(x) = x^3 + 2x^2 + \sin(x)$$

$$f' = 3x^2 + 4x + \cos x$$

$$f'' = 6x + 4 - \sin(x)$$

$$f''' = 6 - \cos(x)$$

$$f'''' = -\sin(x)$$

Parametrical form of functions.

$$y = y(t) \qquad \frac{dy}{dx} = \frac{y'(t)\,dt}{x'(t)\,dt} = \frac{y'(t)}{x'(t)},$$
$$x = x(t)$$

$$y'' = \left(\frac{y'(t)}{x'(t)}\right)' = \frac{y''(t)x'(t) - y'(t)x''(t)}{(x'(t))^2}$$

Example: $y = t\cos(t)$, $x = \sin(t)$

$$\frac{d^2y}{dx^2} = \frac{d}{dy}\left(\frac{\cos(t) - t\sin(t)}{\cos(t)}\right) =$$

$$= \frac{(2\sin(t) - t\cos t)\cos(t) + \sin(t)(\cos(t) - t\sin(t))}{\cos^2(t)} = \frac{3\sin(t)\cos(t) - t}{\cos^2(t)}$$