Lecture 4

Derivatives

O. M. Kiselev,

Innopolis University,

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If $\dfrac{f(x)}{g(x)} \to 0$, $x \to x_0$, we will write:

$$f(x) = o(g(x)), \quad x \to x_0$$

and we will say:

„f is of order less than g"

briefly: „f" equals „o" small of g

Examples:

$$\sin(x) = o(1), \quad x \to 0$$

$$(x - x_0)^2 = o((x - x_0)) \quad x \to x_0$$

$$x - a = o(\sqrt{x-a}) \quad x \to a+0.$$

If $\dfrac{f(x)}{g(x)} = 1$, $x \to x_0$

we will write $f(x) \sim g(x)$, $x \to x_0$

and will say:

"f" is asymptotic to "g"

or $g(x)$ asymptotic approximates $f(x)$

examples: $x^2 + x \sim x$, $x \to 0$

$\sin(x) \sim x$, $x \to 0$

$x + \frac{1}{x} \sim \frac{1}{x}$, $x \to 0$.

Theorem. $f(x) \sim g(x)$, $x \to x_0$, $f(x_0) \neq 0$, $g(x_0) \neq 0$

is necessary and sufficient:

$$f(x) - g(x) = o(g(x)) \quad \text{or} \quad g(x) - f(x) = o(f(x)).$$

Proof. Let $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = 1 \Rightarrow \lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} - 1 = 0$

then $\lim\limits_{x \to x_0} \dfrac{f(x) - g(x)}{g(x)} = 0 \Rightarrow f(x) - g(x) = o(g(x))$

sufficiency: Let $g(x) - f(x) = o(g(x)) \Rightarrow$

$$1 - \frac{f(x)}{g(x)} = \frac{o(g(x))}{g(x)} \Rightarrow$$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$$

If $g \sim f$, $f \sim h \Rightarrow g \sim h$, $x \to x_0$

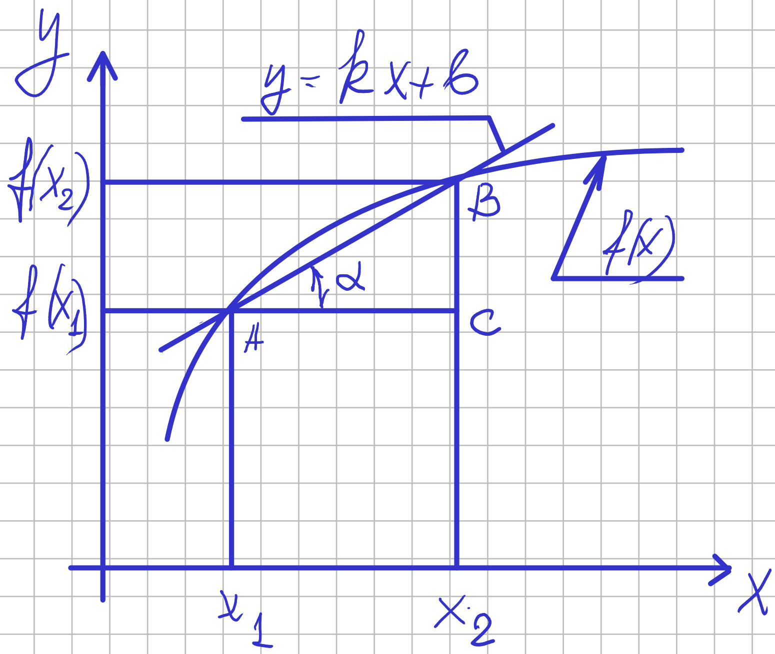
$$\lim_{x \to x_0} \frac{g(x)}{f(x)} \cdot \lim_{x \to x_0} \frac{f(x)}{h(x)} = \lim_{x \to x_0} \frac{g(x)}{f(x)} \cdot \frac{f(x)}{h(x)} = \lim_{x \to x_0} \frac{g(x)}{h(x)}$$

Usage of the asymptotic properties.

1) $$\lim_{x\to 0} \frac{3x+x^3}{2x+5x^2} = \lim_{x\to 0} \frac{x\cdot(3+x^2)}{x(2+5x)} = \lim_{x\to 0} \frac{3+x^2}{2+5x} = \frac{3}{2}$$

2) $$\lim_{x\to 1} \frac{\sin(\ln(x))}{x-1} = \lim_{y\to 0}\left(\frac{\sin(\ln(1+y))}{y}\right) =$$

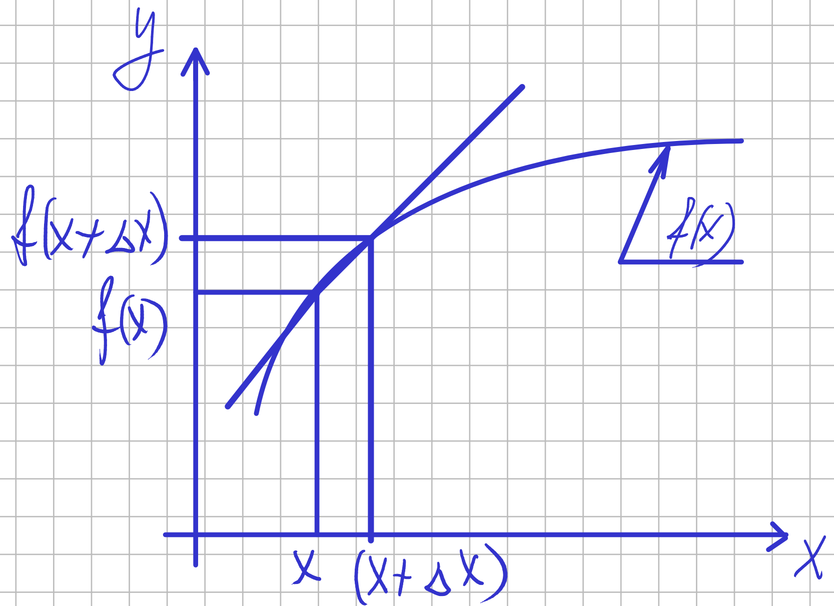
$$= \lim_{y\to 0}\left(\frac{\sin(y+o(y))}{y}\right) = 1.$$



$$k = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$k = \tan(\alpha)$$

$$y - f(x_1) = k(x - x_1)$$



For a tangent line

$$k = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Let's call such limit : $\lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

by the derivative of function $f(x)$ at the point x

and denote:

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x)$$

<u>Examples</u> 1) Let $x(t)$ is a coordinate of a particle on a straight line.

$x(t+\Delta t)$ - a coordinate at a moment $(t+\Delta t)$.

$\dfrac{x(t+\Delta t)-x(t)}{\Delta t}$ - an average speed on interval $(t, t+\Delta t)$;

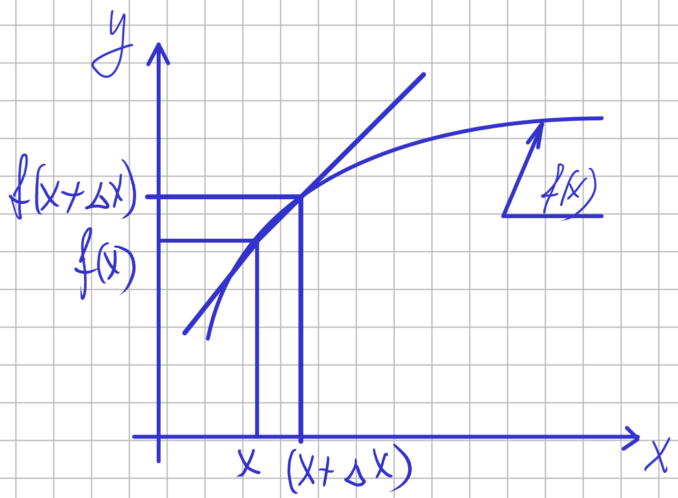
$\lim\limits_{\Delta t\to 0} \dfrac{x(t+\Delta t)-x(t)}{\Delta t} = x'(t)$ - an instantaneous speed at the moment.

2) Let $Q(t)$ is a charge of a battery.

$\frac{Q(t+\Delta t) - Q(t)}{\Delta t}$ is average current through the battery

$Q'(t)$ is an instanteneous current at the moment t.

Differential of a function



Locally one can use a tangent line as an approximation of the $f(x)$:

$$f(x+\Delta x) = f(x) + \underline{f'(x)\Delta x} + o(\Delta x)$$

<u>linear part of variation</u> of the function $f(x)$:

$\underline{f'(x)\Delta x}$ is called a <u>differential of $f(x)$</u>

$$f'(x)\Delta x \equiv df \equiv df(x).$$

The same for $f(x) \equiv x \Rightarrow df(x) \equiv dx$

a differential of independent variable

Another notations for the derivative.

Let denote $f(x+\Delta x)-f(x)=\Delta f$, then the definition of the derivative we can write:

$$\lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x}=f'(x).$$

The same using differentials one can write

$$\lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x}=\frac{df}{dx}$$

Following to I. Newton the derivative of t denotes as $\dot{x}$ which is $\frac{dx(t)}{dt}$ in mechanics.

Examples of derivatives

1) <u>a trivial case</u> $f = C$, $C = \text{const} \Rightarrow$

$$f\big|_{x=x+\Delta} = C, \quad f\big|_{x=x} = C \Rightarrow \Delta f = 0$$

$$\lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = 0. \Rightarrow C' = 0.$$

2) <u>$f(x) \equiv x$</u> $\Rightarrow \displaystyle\lim_{\Delta x \to 0} \frac{(x+\Delta x) - x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$

3) $f(x) = x^2 \Rightarrow \displaystyle\lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cancel{x^2} + 2x\Delta x + \Delta x^2 \cancel{- x^2}}{\Delta x} =$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x} = 2x$$

$$(x^2)' = 2x.$$

4) $$\left(\sqrt{x}\right)' = \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} = \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}} =$$

$$= \frac{(x+\Delta x) - x}{\Delta x} \cdot \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}},$$ then:

$$\lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

5) $$(e^x)' = \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} = \lim_{\Delta x \to 0} e^x \frac{e^{\Delta x} - 1}{\Delta x} =$$

$$= e^x \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = \left| \Delta x = \log(1+y) \right| =$$

$$= e^x \lim_{y \to 0} \frac{e^{\log(1+y)} - 1}{\log(1+y)} = e^x \lim_{y \to 0} \frac{1+y-1}{\log(1+y)} =$$

$$= e^x \lim_{y \to 0} \frac{y}{\log(1+y)} = e^x \lim_{y \to 0} \frac{1}{\log\left((1+y)^{1/y}\right)} =$$

$$= e^x \frac{1}{\log\left(\lim_{y \to 0} (1+y)^{1/y}\right)} = e^x \cdot \frac{1}{\log(e)} = e^x$$

6) $$\left(\log(x)\right)' = \lim_{\Delta x \to 0} \frac{\log(x+\Delta x) - \log(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\log\left(1+\frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}\cdot x} =$$

$$= \frac{1}{x} \lim_{\Delta x \to 0} \log\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}} = \left| \frac{\Delta x}{x} = t \right| =$$

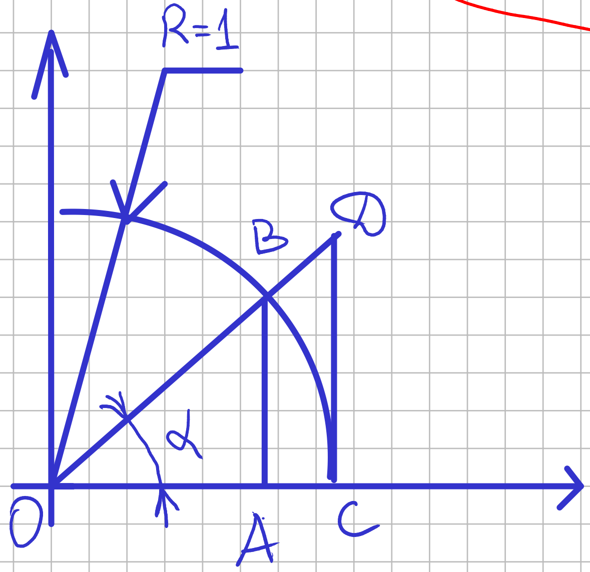
$$= \frac{1}{x} \lim_{t \to 0} \log\left(1+t\right)^{1/t} = \frac{1}{x} \log e = \frac{1}{x}$$

$$\left(\log(x)\right)' = \frac{1}{x}$$

7) $\left(\sin(x)\right)' = \lim\limits_{\Delta x \to 0} \dfrac{\sin(x+\Delta x) - \sin(x)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{2\cos\left(\frac{2x+\Delta x}{2}\right)\sin\left(\frac{\Delta x}{2}\right)}{\Delta x} =$

$$= \lim_{\Delta x \to 0} \cos\left(x + \frac{\Delta x}{2}\right) \cdot \lim_{\Delta x \to 0} \frac{\sin\left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)} =$$

$$= \cos(x) \cdot \lim_{\Delta x \to 0} \frac{\sin\left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)} = \cos(x)$$



$$S_{\triangle OAB} = \frac{1}{2}\sin(\alpha)\cos(\alpha)$$

$$S_{\triangle OCB} = \frac{\alpha}{2}$$

$$S_{\triangle OCD} = \frac{1}{2}\operatorname{tg}\alpha$$

$$S_{OAB} < S_{OCB} < S_{OCD}$$

$$\frac{1}{2}\sin(\alpha)\cos(\alpha) < \frac{\alpha}{2} < \frac{1}{2}\frac{\sin(\alpha)}{\cos(\alpha)}$$

$$\cos(\alpha) < \frac{\alpha}{\sin(\alpha)} < \frac{1}{\cos(\alpha)}, \quad \alpha \to 0$$

$$\frac{\alpha}{\sin(\alpha)} \to 1$$

Differentiable functions

Let $f(x)$ is defined in neighbourhood of x_0 and $f(x_0+\Delta x) = f(x_0) + \underline{A\Delta x} + o(\Delta x)$, where $A = const$.

Such function is called <u>a differentiable function</u> in the point x_0.

Examples: 1) $(x_0+\Delta x)^2 = x_0^2 + 2x_0\Delta x + \Delta x^2$,

here $A = 2x_0 = const$ for $\forall\, x_0 = const$.

2) $\sin(x_0+\Delta x) = \sin(x_0) + \cos(x_0)\Delta x + o(\Delta x)$,

here $A = \cos(x_0)$.

Theorem. If $f(x)$ has a derivative at x_0, then $f(x)$ is differentiable function at x_0 and the differential of the function $df = f'(x_0)dx$.

Proof. $f'(x_0) = \lim\limits_{\Delta x \to 0} \dfrac{f(x_0+\Delta x) - f(x_0)}{\Delta x} \Rightarrow$

$\forall \varepsilon > 0 \ \exists \delta > 0 : \left| \dfrac{f(x_0+\Delta x) - f(x_0)}{\Delta x} - f'(x_0) \right| < \varepsilon$, $\Delta x < \delta$.

$$f(x_0+\Delta x) - f(x_0) = f'(x_0)\Delta x + o(\Delta x).$$

That means $df = f'(x_0)dx$

Theorem. If $f(x)$ is differentiable at x_0 then $f(x)$ is continuous at x_0.

Proof. $f(x)-f(x_0) = A(x-x_0) + o(x-x_0) \Rightarrow$

$$\lim_{x\to x_0} f(x)-f(x_0) = A \lim_{x\to x_0}\left((x-x_0) + o(x-x_0)\right) = 0$$

The controrverse is not true!

$f(x)=\sqrt{|x|}$, $\sqrt{|x|}$ is continuous function at $x=0$:

$\forall \varepsilon>0 \; \exists \delta(\varepsilon)=\varepsilon^2 :$ $\sqrt{|x|}<\varepsilon$, $|x|<\delta$

But $\left(\sqrt{|x|}\right)'$ does not exists at $x=0$; for $x>0$;

$$\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}} \to \infty, \text{ as } x\to 0.$$

Differential rules.

$(f(x)+g(x))' = f'(x)+g'(x)$ – the straightforward corollary from the definition.

$$(f(x)\cdot g(x))' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)\,g(x+\Delta x) - f(x)g(x)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)\,g(x+\Delta x) + f(x+\Delta x)\,g(x) - f(x+\Delta x)\,g(x) - f(x)\,g(x)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{(f(x+\Delta x) - f(x))\,g(x) + f(x+\Delta x)\,(g(x+\Delta x) - g(x))}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{(f(x+\Delta x) - f(x))}{\Delta x}\,g(x) + \lim_{\Delta x \to 0} \frac{f(x+\Delta x)\,(g(x+\Delta x) - g(x))}{\Delta x} = f'(x)g(x) + f(x)g'(x)$$

Differential rules

$$\left(\frac{1}{f(x)}\right)' = \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left(\frac{1}{f(x+\Delta x)} - \frac{1}{f(x)}\right) = \lim_{\Delta x\to 0}\left(\frac{f(x)-f(x+\Delta x)}{f(x+\Delta x)\,f(x)}\right)\frac{1}{\Delta x} =$$

$$= \lim_{\Delta x\to 0}\frac{1}{f(x+\Delta x)\cdot f(x)}\cdot\lim_{\Delta x\to 0}\frac{f(x)-f(x+\Delta x)}{\Delta x} = \frac{-f'(x)}{f^2(x)}.$$

$$\left(\frac{f(x)}{g(x)}\right)' = \left(f(x)\cdot\frac{1}{g(x)}\right)' = f'(x)\cdot\frac{1}{g(x)} + f(x)\cdot\left(\frac{1}{g(x)}\right)' = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g^2(x)} =$$

$$= \frac{f'(x)\cdot g(x) - f(x)g'(x)}{g^2(x)}.$$

Differential rules

Suppose that $g(x)$ has a derivative at x and $f(y)$ has a derivative at $y = g(x)$, then:

$$f(g(x))' = \lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{f(g(x) + g'\Delta x + o(\Delta x)) - f(g(x))}{\Delta x} \cdot \frac{g'(x)}{g'(x)} =$$

$$= \lim_{\Delta x \to 0} \frac{f(g(x) + g'(x)\Delta x + o(\Delta x)) - f(g(x))}{g'(x)\Delta x} \cdot g'(x) =$$

$$= g'(x) \cdot f'(g(x)).$$

Examples.

$$\left(x^2 \cdot \sin(x)\right)' = \left(x^2\right)' \cdot \sin(x) + x^2 \cdot \left(\sin(x)\right)' =$$
$$= 2x \sin(x) + x^2 \cos(x).$$

$$\left(\sin(x^2)\right)' = \left(x^2\right)' \cdot \sin'(x^2) = 2x \cdot \cos(x^2).$$

$$\left(x^{\alpha}\right)' = \left(e^{\alpha \ln x}\right)' = \alpha \left(\ln(x)\right)' e^{\alpha \ln(x)} =$$

$$= \frac{\alpha}{x} \cdot e^{\alpha \ln(x)} = \frac{\alpha}{x} x^{\alpha} = \alpha x^{\alpha - 1}$$

$$\left(\operatorname{tg} x\right)' = \left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{\left(\sin(x)\right)' \cdot \cos(x) - \sin(x) \cdot \left(\cos(x)\right)'}{\cos^2(x)} =$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}.$$

An implicit differentiation:

The implicit form for the function $y(x)$: $\Phi(x, y) = 0$.

1) Implicit form of function:

$$y^2 + x^2 = 1 \Rightarrow y(x) = \begin{cases} +\sqrt{1-x^2}, & y \geq 0; \\ -\sqrt{1-x^2}, & y < 0. \end{cases}$$

Let's differetiate the formula $y^2 + x^2 = 1$

$$2\, y' \cdot y(x) + 2x = 0 \Rightarrow y' = -\frac{x}{y},$$

but: $y^2 + x^2 = 1 \Rightarrow (y')^2 = \frac{x^2}{y^2} = \frac{x^2}{1-x^2} \Rightarrow$

$$(y')^2 = \frac{x^2}{1-x^2}$$

2) $x^3 + 4y^3 + 2x = 0$

$3x^2 + 12y^2y' + 2 = 0$

$y' = -\dfrac{2+3x^2}{12y^2}$, $\quad 4y^3 = -2x - x^3$,

$\dfrac{y'}{y} = \dfrac{2+3x^2}{-3(4y^3)} = \dfrac{2+3x^2}{3(2x-x^3)}$

$y' = \dfrac{2+3x^2}{3(2x-x^3)} \cdot y \Rightarrow y' = \dfrac{2+3x^2}{3(2x-x^3)} \sqrt[3]{\dfrac{-2x-x^3}{4}}$.

Guidlines for differentiation of implicit function

1) Differentiate the formula with respect to x (or y if you wish)

2) Collect all terms which contain $\left(\frac{dy}{dx}\right)$ or $\left(\frac{dx}{dy}\right)$.

3) Try to use obtained formula for defining the derivative.

Resume

1. Definition of a derivative
2. Differentiable functions
3. Derivatives of elementary functions.
4. Differentiation rules.
5. Differentiation of implicit forms of functions