Fundamental theorem of calculus

O.M. Kiselev
o.kiselev@innopolis.ru

Innopolis university

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On the previous lecture

Geometric sense of the antiderivative

Fundamental theorem of calculus

Definite integral

Definition of an indefinite integral

$$\int f(x)dx \equiv F(x) + C, \quad \forall C \in \mathbb{R}$$

- $\int$ is an integral sign.
- $f(x)$ is called an integrand.
- dx is a differential of the variable of integration.
- $F(x)$ is an antiderivative of the integrand.
- C is an arbitrary constant.

Theorem about integration by substitution

Theorem

Let $F(x)$ and $G(x)$ are differentiable functions and range of $G(x)$ in domain of the $F(x)$. Then

$$\int F'(G(x))G'(x)dx = F(G(x)) + C.$$

Proof. Differentiate this formula straightforward!
The same formula:

$$\int F'(G(x))\,dG(x) = F(G(x)) + C.$$

An example

$$\int \sin(\cos(x))\sin(x)dx = -\int \sin(\cos(x))d(\cos(x) =$$

$$|t = \cos(x)| =$$

$$-\int \sin(t)\,dt = \cos(t) + C = \cos(\cos(x)) + C.$$

An integration by parts

Theorem about integration by parts

If the function $u(x)$ and $v(x)$ are differentiable on some interval E, then on the interval the following formula is valid:

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx.$$

Proof. Differentiate the formula! The same formula:

$$\int u(x)\,dv(x) = u(x)v(x) - \int v(x)\,du(x).$$

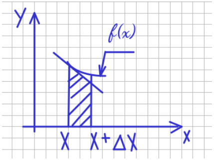
An example

$$\int xe^{\lambda x}dx = \int xd\left(\frac{e^{\lambda x}}{\lambda}\right) =$$

$$x\,\frac{e^{\lambda x}}{\lambda} - \frac{1}{\lambda}\int e^{\lambda x}\,dx =$$

$$x\,\frac{e^{\lambda x}}{\lambda} - \frac{e^{\lambda x}}{\lambda^2} + C.$$

Geometry of the antiderivative



Let $F(x)$ is an the antiderivative of $f(x)$:

$$f(x) = \frac{F(x+\Delta x) - F(x))}{\Delta x},\ \Delta x \to 0.$$

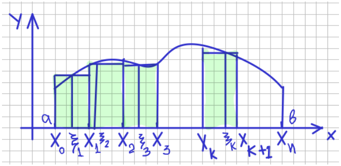
Then:

$$F(x+\Delta x) - F(x) = f(x)\Delta x,\ \Delta x \to 0.$$

Let's use the Lagrange theorem on the interval $[x, x+\Delta x]$:

$$F(x+\Delta x) - F(x) = f(\xi)\Delta x.$$

The elementary area under the curve



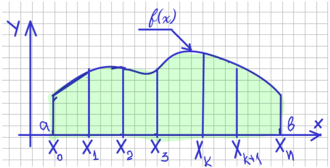
Consider an interval $[a, b]$ and suppose $f(x)$ is continuous function on this interval. Let $\Delta_k > 0$ is a k-th step for partition of the interval:

$$x_0 = a,\ x_1 = x_0 + \Delta_1,\ \ldots, x_{k+1} = x_k + \Delta_k, \ldots, x_n = b.$$

In the simplest case the all might be the same $\Delta_k = \Delta$. Then

$$s_k = f(\xi_k)\Delta_k = F(x_{k-1}) - F(x_k)$$

The sum as a difference of antiderivatives



Let's consider a sum:

$$\sum_{k=1}^{n} f(\xi_k)\Delta_k = \sum_{k=1}^{n}(F(x_k) - F(x_{k-1})) = F(x_n) - F(x_0),$$

or

$$\sum_{k=1}^{n} f(\xi_k)\Delta_k = F(b) - F(a).$$

$$\lim_{\max \Delta_k \to 0} \sum_{k=1}^{n} f(\xi_k)\Delta_k = F(b) - F(a).$$

A definite integral

$$\lim_{\max_k \Delta_k \to 0} \sum_{k=1}^{n} f(\xi_k)\Delta_k = \int_a^b f(x)dx.$$

Fundamental theorem of calculus

If the function f(x) has an antiderivative $F(x)$ on the interval $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

The formula is called as Newton-Leibniz formula.

A usage of the Newton-Leibniz formula

$$\begin{aligned}
\int_0^1 x dx &= \frac{x^2}{2}\Big|_{x=0}^{x=1} = \frac{1}{2}. \\
\int_{-\pi/2}^{\pi/2} \sin(x) dx &= -\cos(x)|_{x=-\pi/1}^{x=\pi/2} \\
&= -\cos(\pi/2) + \cos(-\pi/2) = 0. \\
\int_1^{e^5} \frac{dx}{x} &= \log(x)|_{x=1}^{x=e^5} = 5.
\end{aligned}$$

Integration by substitution

Let $F(x)$ be an anitderivative of $f(x)$, then a general rule for integration:

$$\begin{aligned}\int_a^b f(G(x))G'(x)dx &= \int_{G(a)}^{G(b)} f(G(x))dG(x) \\ &= F(G(b)) - F(G(a)).\end{aligned}$$

Integration by substitution

A general rule:

$$\int_a^b f(x)dx = \left| \begin{array}{c} x = g(t), \\ dx = g'(t)dt \end{array} \right| = \int_\alpha^\beta f(g(t))g'(t)dt.$$

Here limits of integration α and β are such that
$g(\alpha) = a$, $g(\beta) = b$.
As a result one gets:

$$\int_a^b f(x)dx = \int_\alpha^\beta h(t)dt,\ h(t) = f(g(t))g'(t).$$

Integration by substitution. An example

$$\int_0^1 \sqrt{1-x^2}dx = \left| \begin{array}{c} x = sin(t),\ dx = \cos(t)dt \\ \alpha = \arcsin(0) = 0,\ \beta = \arcsin(1) = \pi/2 \end{array} \right| =$$

$$\int_0^{\pi/2} \sqrt{1-\sin^2(t)}\cos(t)dt = \int_0^{\pi/2} \cos^2(t)dt$$

$$= \int_0^{\pi/2} \frac{1+\cos(2t)}{2}dt =$$

$$\int_0^{\pi/2} \frac{dt}{2} + \int_0^{\pi/2} \frac{\cos(2t)}{2}\frac{d(2t)}{2} =$$

$$\frac{\pi}{4} + \frac{1}{4}(\sin(\pi) - \sin(0)) = \frac{\pi}{4}.$$

Integration by parts

A general rule:

$$\int_a^b u(x)v'(x)dx = \int_a^b u(x)dv(x) =$$

$$u(b)v(b) - u(a)v(a) - \int_a^b v(x)u'(x)dx.$$

An example:

$$\int_0^{2\pi} x\sin(x)dx = -\int_0^{2\pi} xd\cos(x) =$$

$$-x\cos(x)|_{x=0}^{x=2\pi} + \int_0^{2\pi}\cos(x)dx =$$

$$-2\pi\cos(2\pi) + \sin(x)|_{x=0}^{x=2\pi} = -2\pi.$$

Riemann sum

Consider a partition of the interval $[a, b]$:

$a < x_1 < x_2 < \cdots < x_k < x_{k+1} < \cdots < x_{n-1} < b,\ \Delta x_k = x_{k+1} - x_1.$

The sum

$$S_n = \sum_{k=1}^{n} f(\xi_k)\Delta x_k,\ \xi_k \in [x_k, x_{k+1}]$$

is called Riemann sum.

Definite integral as a limit of Riemann sum

Definition. The function $f(x)$ is called integrable

on the interval $[a, b]$ if there exists a limit of Remanian sum

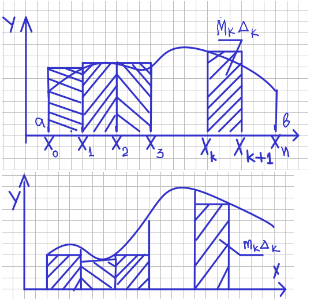
$$A = \lim_{\max \Delta_k \to 0} \sum_{k=1}^{n} f(\xi_k)\Delta x_k,$$

$\forall \{x_k\}_{k=1}^{n}$ and $\forall \xi_k \in [x_k, x_{k+1}]$

By another words: there exists A such that for any $\epsilon > 0$ exists $\delta > 0$ such that

$$|A - \sum_{k=1}^{n} f(\xi_k)\Delta x_k| < \epsilon, \max_k \Delta_k < \delta, \forall \xi \in [x_k, x_{k+1}].$$

Darboux sums



Let's define
an upper Darboux sum:

$$S_n = \sum_{k=1}^{n} M_k \Delta_k,$$

$$M_k = \max_{x \in [x_k, x_{k+1}} (f(x)),$$

and a lower Darboux sum:

$$s_n = \sum_{k=1}^{n} m_k \Delta_k,$$

$$m_k = \min_{x \in [x_k, x_{k+1}} (f(x)).$$

Properties of the Darboux sums

$\forall$ partitions $X = \{x_k\}_{k=1}^{n}$ and $X' = \{x'_k\}_{k=1}^{n'}$
$s_n < S_{n'}$.

Proof. Let's construct partition $\tilde{X}$ which contains both X and X'. For any intersected intervals of X and X': $m_k \leq M_{k'}$, then

$$s_n = \sum_{\tilde{X}} m_k \tilde{\Delta} \tilde{x}_k \leq \sum_{\tilde{X}} M_{k'} \tilde{\Delta} \tilde{x}_k = S_{n'}.$$

Darboux and Riemann sums

$\forall$ partition $X = \{x_k\}_{k=1}^{n}$:

$$s_n = \sum_{k=1}^{n} m_k \Delta_k \leq \sum_{k=1}^{n} f(\xi_k)\Delta x_k \leq \sum_{k=1}^{n} M_k \Delta_k = S_n.$$

Proof. Let's consider any terms of this sums for certain interval:

$$m_k \Delta x_k \leq f(\xi_k)\Delta x_k \leq M_k \Delta x_k.$$

Then the same is true for the sums.

Continuity implies integrability on a finite interval

Theorem

Let $f(x)$ be continuous on the interval $x \in [a, b]$, then $f(x)$ in integrable on the interval.

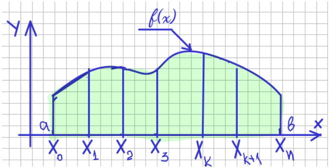
Proof. The continuous function is bounded on the finite interval therefore upper and lower Darboux sums are bounded. Fined the difference on some partition:

$$S_n - s_n = \sum_{k=1}^{n} (M_k - m_k)\Delta x_k \le \max_k (M_k - m_k)(b - a)$$

Due to continuity of $f(x)$ $\forall \epsilon > 0$
$\exists \delta : \max_k (M_k - m_k) < \epsilon/(b - a)$ as $\max_k \Delta_k < \delta$.
Then

$$\lim_{\max_k \Delta x_k \to 0} (S_n - s_n) = 0.$$

Continuity implies integrability on a finite interval



Then

$$\lim_{\max_k \Delta x_k \to 0} S_n = \lim_{\max_k \Delta x_k \to 0} s_n = I,$$

due to the inequality:

$$s_n \leq \sum_{k=1}^{n} f(\xi_k)\Delta x_k \leq S_n.$$

It yields:

$$I = \lim_{\max_k \Delta x_k \to 0} \sum_{k=1}^{n} f(\xi_k)\Delta x_k,$$

or

$$\int_a^b f(x)dx = I.$$

Summary

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