Lecture 3.

Continuous functions.

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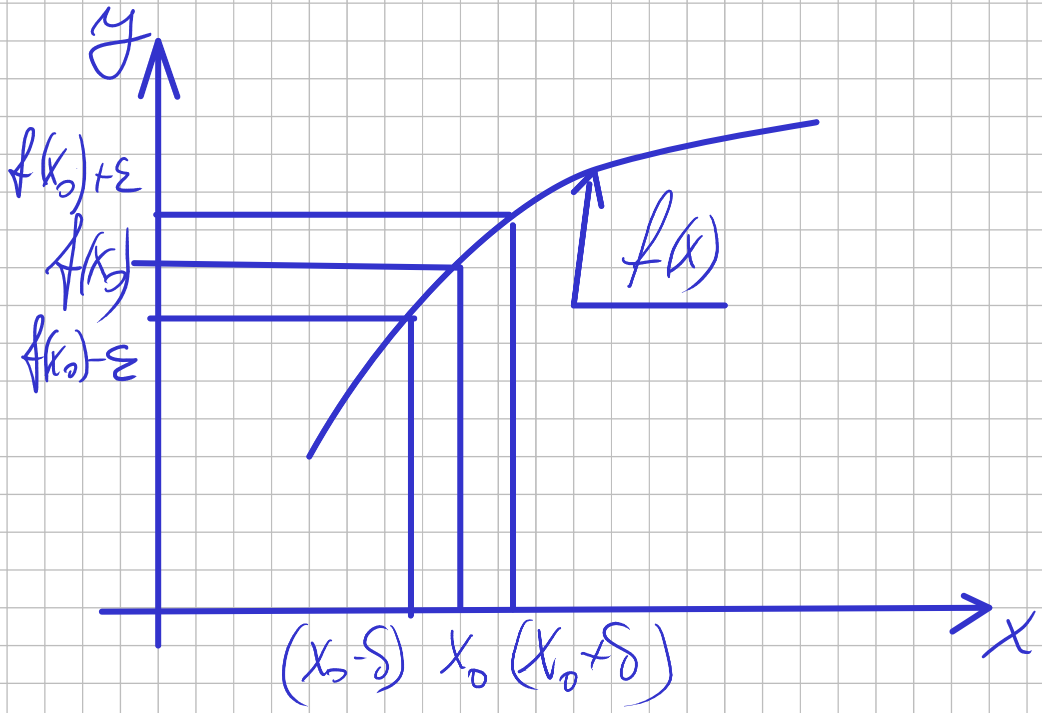


Definition. A function $f(x)$ is called continuous function in a point x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Note. $\lim_{x \to x_0} f(x) = f(x_0)$ means that $f(x)$ is defined function in the point x_0

Trivial examples: i) $y = x^2$ is continuos function for $x \in [-1, 1]$.

Proof. Let $x_0 \in [-1, 1]$, note that $\forall x \in [-1, 1]$:

$$|x^2 - x_0^2| = |(x + x_0) \cdot (x - x_0)| \leq 2|x - x_0|$$

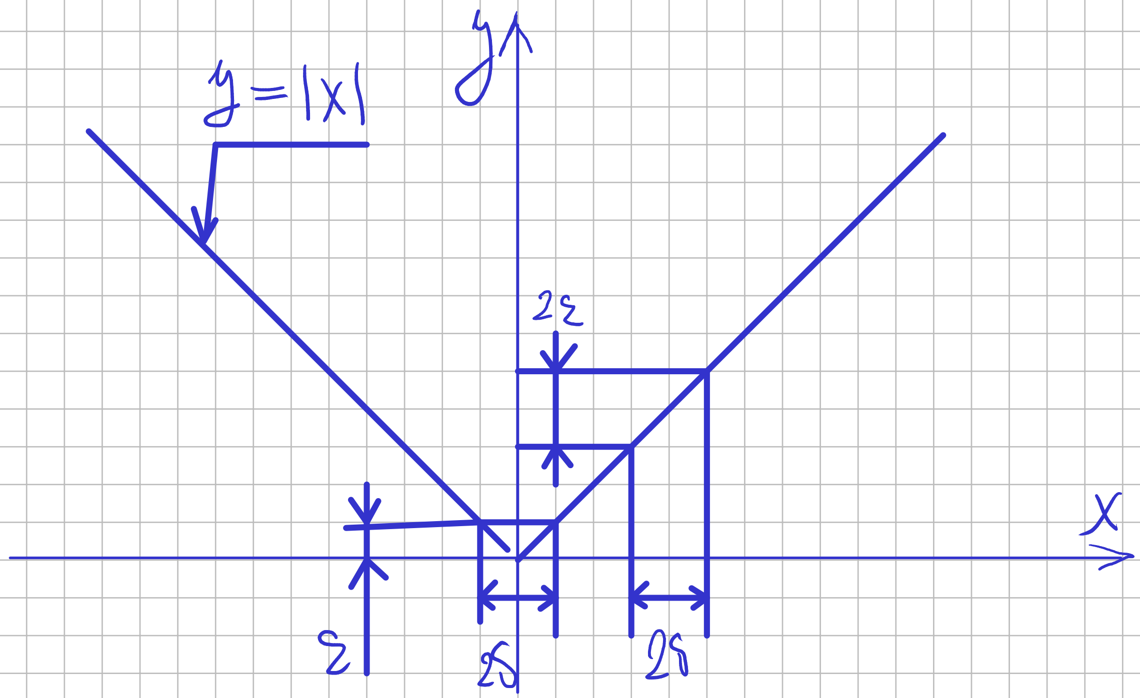
$\forall \varepsilon > 0 \ \exists \delta < \varepsilon/2$ for $\forall x \in [x_0 - \delta, x_0 + \delta]$

$$|x^2 - x_0^2| < \varepsilon.$$

2) $y = |x|$, let $x_0 = 0$, then $\forall \varepsilon > 0$,

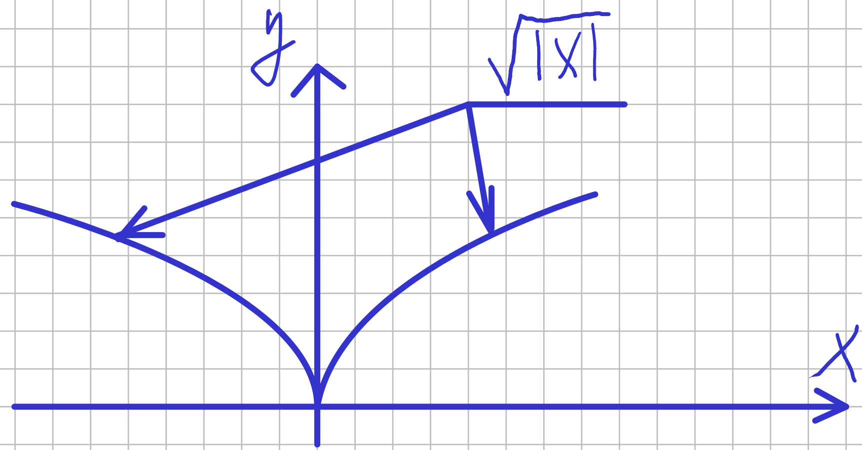
$\delta = \varepsilon$: $x \in (-\varepsilon, \varepsilon)$:

$|x| < \varepsilon \Rightarrow \lim\limits_{x \to 0} |x| = |0| = 0$.



Nontrivial examples

(1) $y=\sqrt{|x|}$.



Let's consider $x_0=0$.

$\Delta y = \sqrt{|x|}$, $\delta(\varepsilon)=\varepsilon^2$, $\Rightarrow$ $\Delta y = \varepsilon \Rightarrow$

$\forall \varepsilon>0$, $\delta(\varepsilon)=\varepsilon^2$ $\forall x \in (-\varepsilon^2, \varepsilon^2)$

$$\sqrt{|x|} < \varepsilon \Rightarrow$$

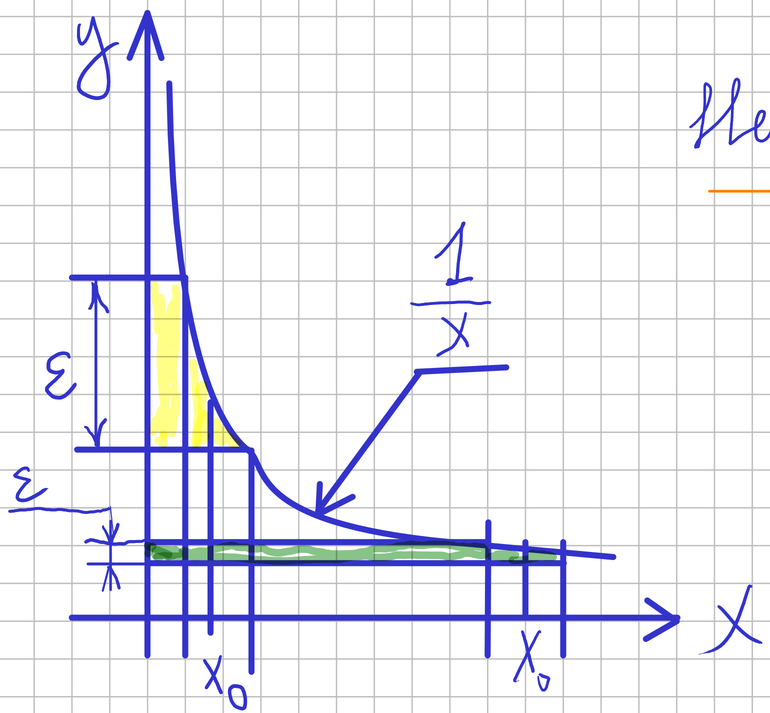
$$\lim_{x\to 0}\sqrt{|x|} = \sqrt{|0|} = 0.$$

(2) $y = \frac{1}{x}$, $x_0 > 0$

Let's consider $y(x_0 + \Delta x) - y(x_0)$:

$$\frac{1}{x_0+\Delta x} - \frac{1}{x_0} = \frac{x_0 - x_0 - \Delta x}{x_0(x_0+\Delta x)} = \frac{-\Delta x}{x_0(x_0+\Delta x)}$$

$$\forall \varepsilon > 0 \ \exists \delta(\varepsilon) : |\Delta x| < \delta \quad \left| \frac{-\Delta x}{x_0(x_0+\Delta x)} \right| < \varepsilon$$



Here $\delta(\varepsilon) < \varepsilon \cdot x_0^2$.

$y = \frac{1}{x}$ is continuous function

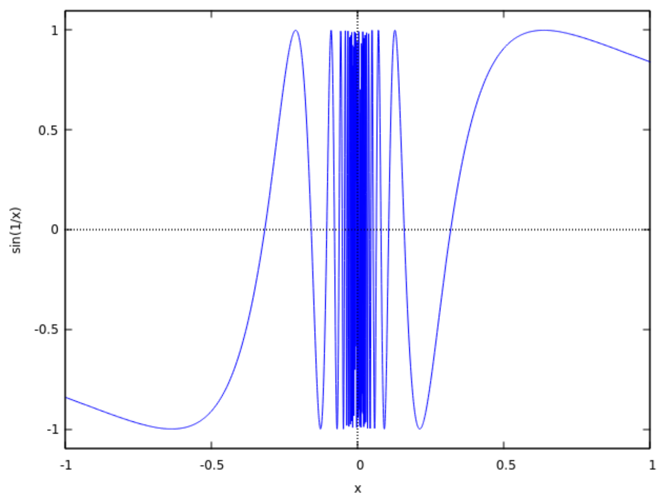
$x = 0$ does not belong to the implied domain for $\frac{1}{x}$!

3) $\sin\left(\frac{1}{x}\right)$ is continuous.

$$f(y) = \sin(y), \quad g(x) = \frac{1}{x}$$

$$\sin\left(\frac{1}{x}\right) = f(g(x))$$

$x = 0$ does not belong to implied domain of $\sin\left(\frac{1}{x}\right)$.



Resume: continuity

$f(x)$ continuous function at a point x_0:

1) $f(x_0)$ is defined.

2) $\lim\limits_{x \to x_0} f(x)$ exists.

3) $\lim\limits_{x \to x_0} f(x) = f(x_0)$

Properties of continuous functions.

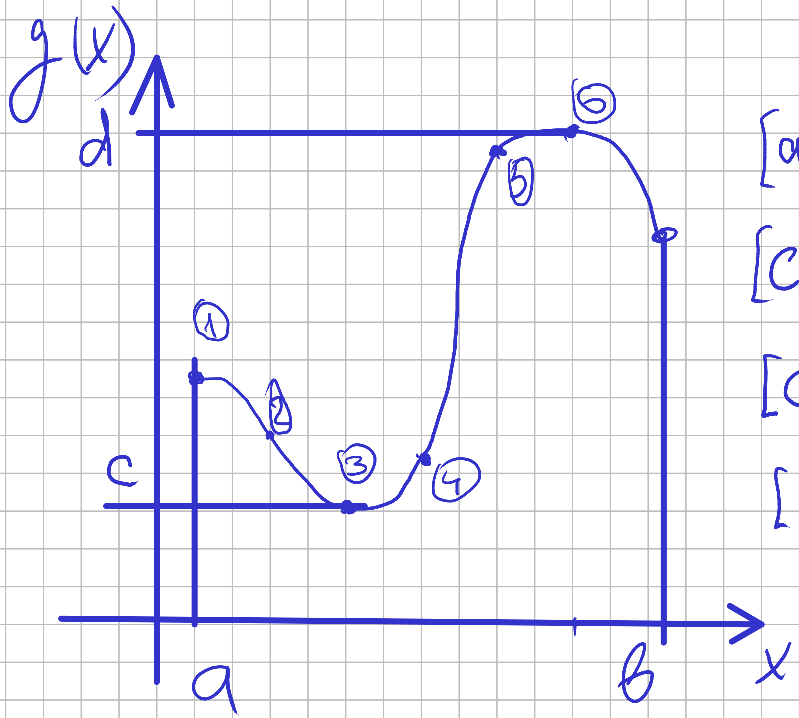
Let $f(x)$, $g(x)$ are continuous function.

$$h(x) = f(x) + g(x),$$

$$h(x) = f(x) \cdot g(x),$$

$$h(x) = f(g(x))$$ if $g(x_0) = y_0$ and $f(y_0)$ are points of continuity

These properties are straightforward corollaries of the same properties for the limits of sequences and definition of a limit of function as a limit of any convergent sequence.

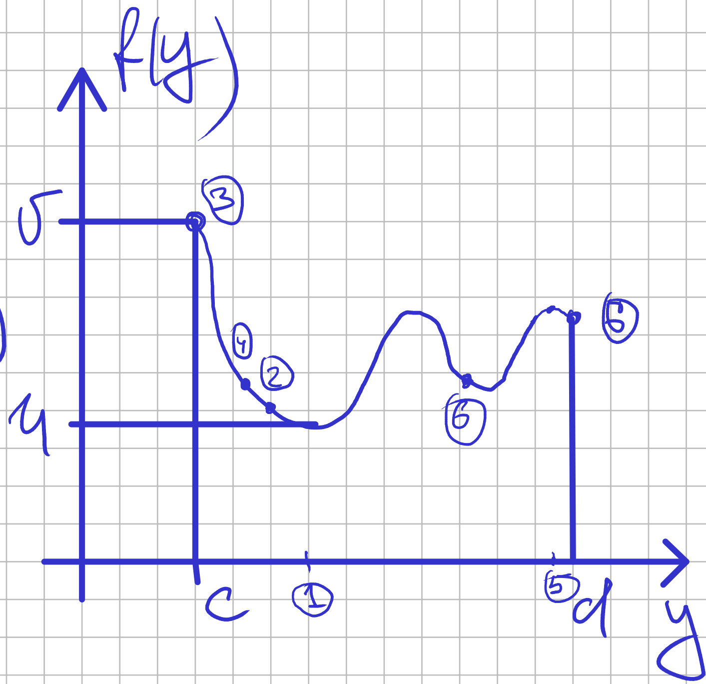


$[a, b]$ is domain of $g(x)$

$[c, d]$ is range of $g(x)$

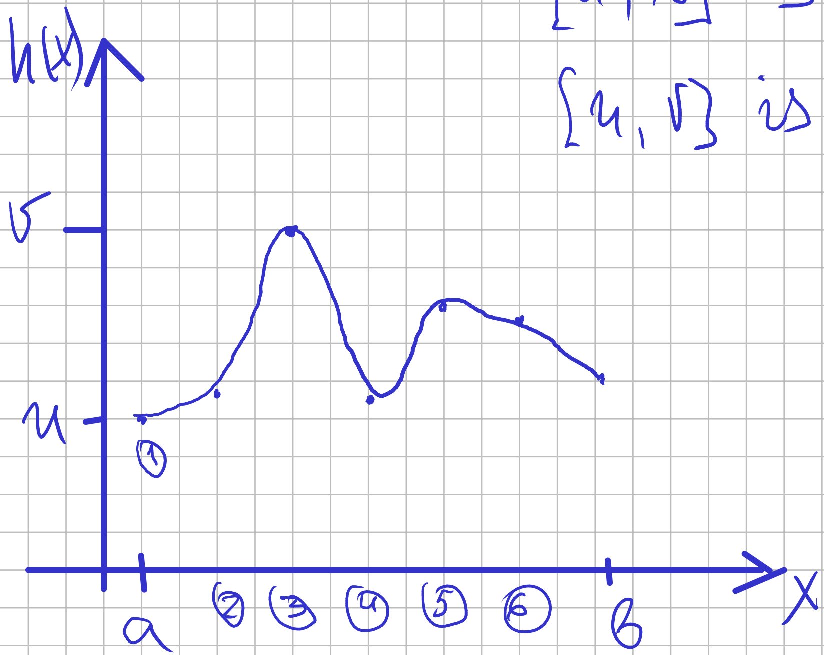
$[c, d]$ is domain of $f(y)$

$[u, v]$ is range of $f(y)$



$[a, b]$ is the domain of $h(x) \equiv f(g(x))$.

$[u, v]$ is the range of $h(x)$.



Let's proove: $h(x) \equiv f(g(x))$ is continuous if $f(x)$ and $g(x)$ are continuous functions.

1) $g(x)$ is continuous at $[a, b]$

2) $f(x)$ is continuous at range $g(x)$ for $x \in [a, b]$

Then: 1) $\forall \sigma > 0 \ \exists \delta(\sigma) > 0, \ x \in (x_0 - \delta, x_0 + \delta), \ |g(x) - g(x_0)| < \sigma$

2) $\forall \varepsilon > 0, \ \exists \sigma > 0, \ y \in (g(x_0) - \sigma, g(x_0) + \sigma), \ |f(y) - f(g(x_0))| < \varepsilon$

This implies:

$$\forall \varepsilon > 0, \ \exists \delta(\varepsilon) > 0, \ x \in (x_0 - \delta, x_0 + \delta): \ |f(g(x)) - f(g(x_0))| < \varepsilon$$

Discontinuity

If $f(x)$ is defined on $[a, b]$ (exept possible x_0, $x_0 \in [a, b]$) and $f(x)$ is not continuos at x_0, then $f(x)$ has discontinuity at x_0

Examples:

$$\operatorname{sign}(x) = \begin{cases} 1, & x > 0; \\ 0, & x = 0; \\ -1, & x < 0. \end{cases}$$

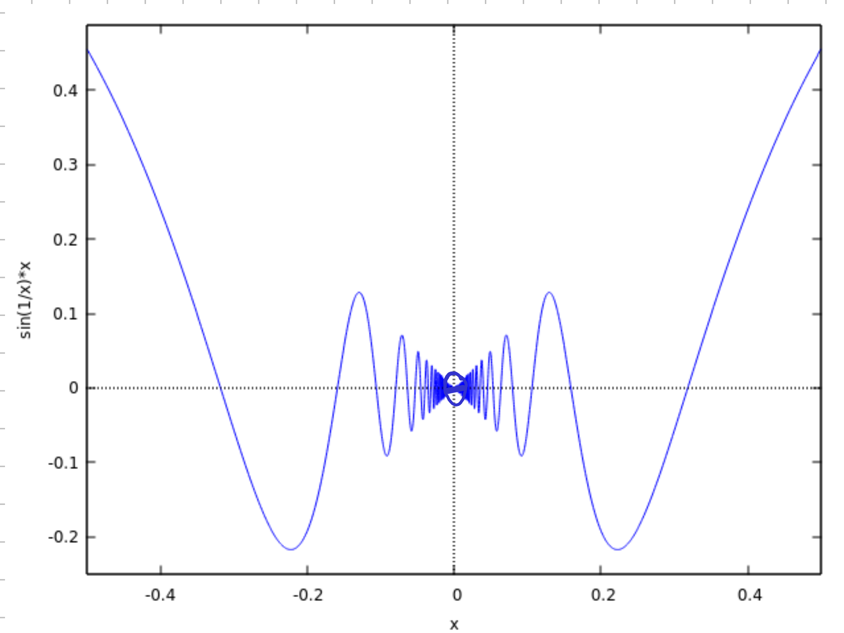
$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0; \\ 0, & x = 0. \end{cases}$$

An example: $x \cdot \sin\left(\frac{1}{x}\right)$ is continuous at $(-\infty, 0) \cup (0, +\infty)$

A counter example:

$$f(x) = \begin{cases} x \cdot \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

$f(x)$ is continuous at $\mathbb{R}$.



If $\lim\limits_{x \to x_0 - 0} f(x) = C$ and $\lim\limits_{x \to x_0 + 0} f(x) = C$,

but $f(x)$ is not defined at $x = x_0$

Then x_0 is removable discontinuity

An example: $x = \sin\left(\frac{1}{x}\right)$, $x \in \mathbb{R} \setminus 0$

$$\lim_{x \to -0} x \sin\left(\frac{1}{x}\right) = 0$$

$$\lim_{x \to +0} x \sin\left(\frac{1}{x}\right) = 0$$

$$f(x) = \begin{cases} x \sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

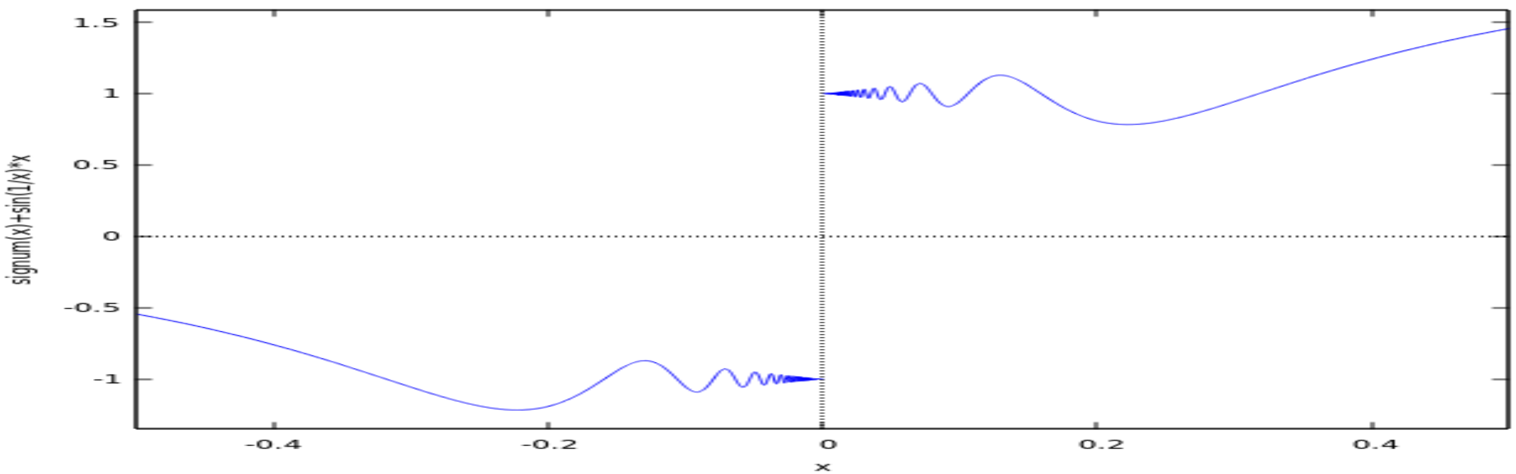
$f(x)$ is continuous for all $x \in \mathbb{R}$

If $\lim_{x \to x_0 - 0} f(x) = A$,

$\lim_{x \to x_0 + 0} f(x) = B$, $\quad A \neq B$

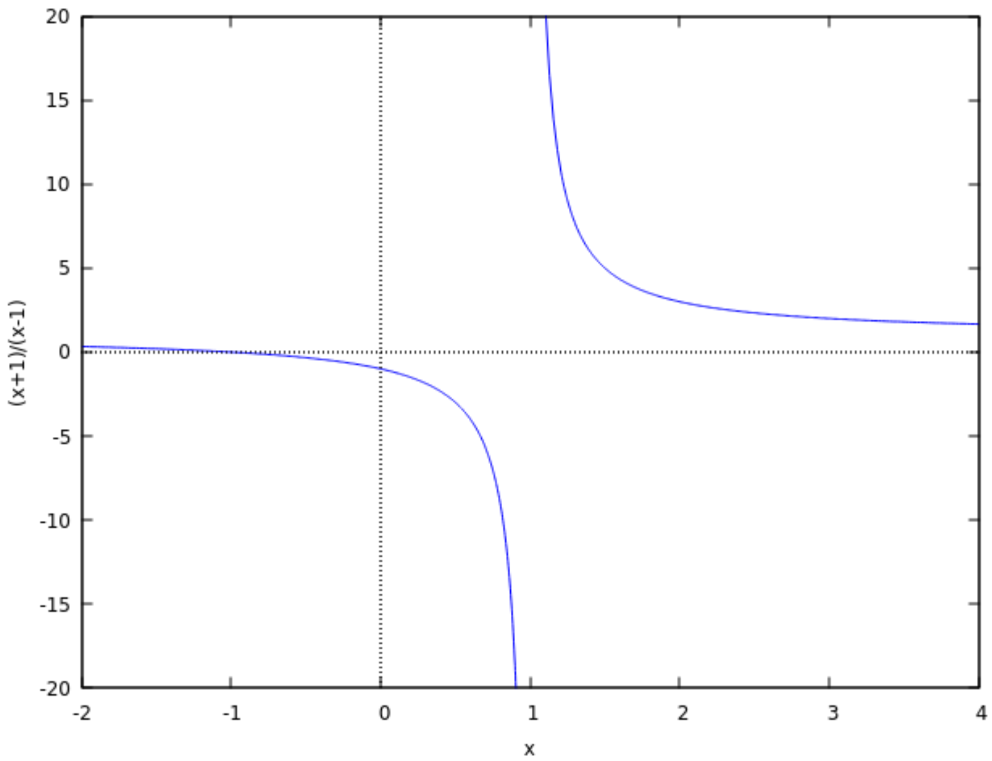
Then $f(x)$ has <u>jump</u> discontinuity

<u>an ex.</u> $f(x) = \operatorname{sign}(x) + x \sin\left(\frac{1}{x}\right)$



If $\lim\limits_{x \to x_0} f(x) = \infty$, then $f(x)$ has infinite discontinuity

An example $f(x) = \dfrac{x+1}{x-1}$



If $f(x)$ as $x \to x_0$ contains bounded divergent sequence $\{f(x_n)\}_{n=1}^{\infty}$, as $\{x_n\}_{n=1}^{\infty} \to x_0$, then $f(x)$ has oscillating discontinuity.

An example: $f(x) = \sin\left(\frac{1}{x}\right)$, at $x_0 = 0$;
$f(x) = \sin(x)$, at $x = \infty$.

Theorem. If $f(x)$ is continuous on $[a,b]$ then $f(x)$ is bounded on $[a,b]$.

Proof. Assume that $\exists$ continuous $f(x)$, $\forall x \in [a,b]$ and $f(x)$ is unbounded.

This means $\exists\ x_0 \in [a,b]$ and $\{x_n\} \to x_0$ such that $f(x_n) \to \infty$, $n \to \infty$, but it contradicts to the condition of the theorem ($f(x)$ has a limit $\forall x \in [a,b]$).

__Theorem.__ $\forall$ continuos function on $[a,b]$ achieves maximal and minimal values at $[a,b]$.

__Proof:__ Assume $\exists \sup_{x\in[a,b]}\{f(x)\}=A$, but $\not\exists\, x_0\ f(x_0)=A$

$x_0\in[a,b]$. Define $\varphi(x)=\dfrac{1}{A-f(x)}$, and $f(x)-A\neq 0$,

$\varphi(x)$ is continuous,

but $\forall \varepsilon>0\ \exists x: |A-f(x)|<\varepsilon, \Rightarrow \varphi(x)>\frac{1}{\varepsilon} \Rightarrow$

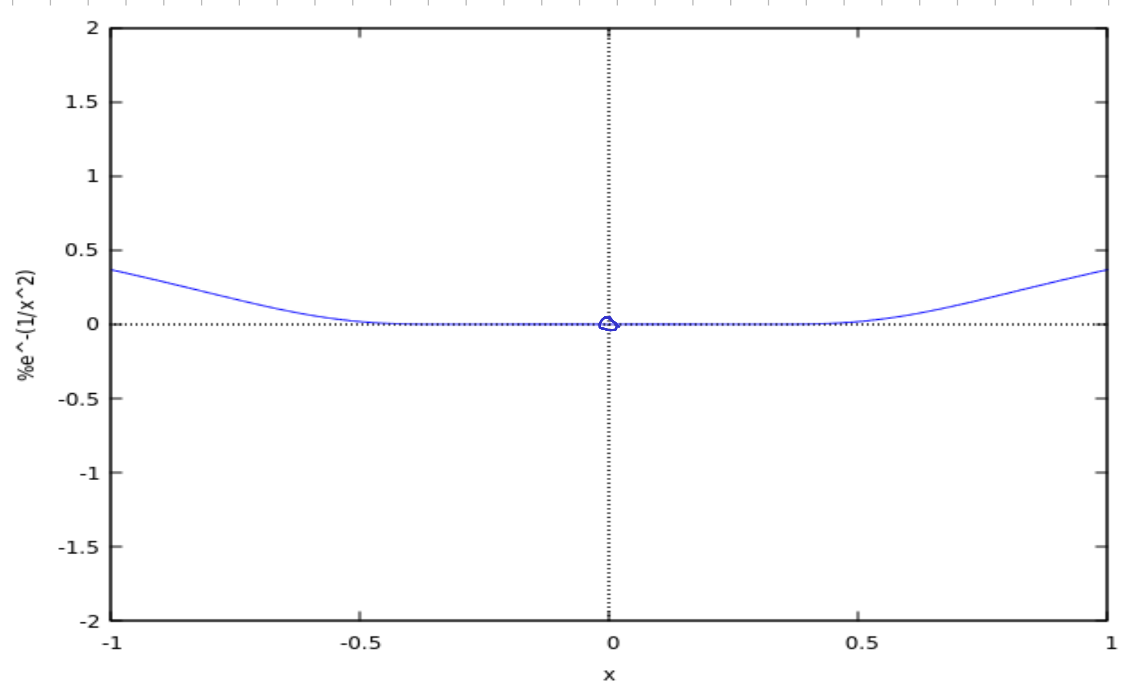
$\varphi(x)$ is unbounded. We obtain a contradiction to Theorem about boundedness of continuos function. $\Rightarrow \exists x: f(x)=A. \Rightarrow A=\max_{x\in[a,b]} f(x)$.

Counterexamples.

1). $\dfrac{1}{1+x^2}$, $x \in \mathbb{R}$: $x \in (-\infty, +\infty)$,

2) $e^{-\frac{1}{x^2}}$ $x \in \mathbb{R} \setminus 0$.

3) $y = x$, $x \in (0, 1)$.



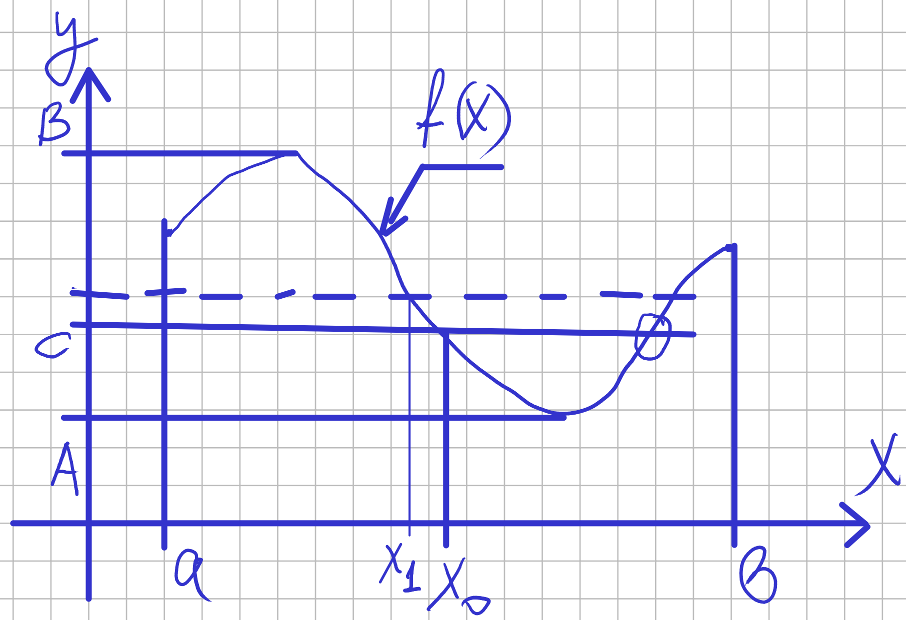
Intermediate value theorem

Let $f(x)$ be continuous on $[a, b]$.

Define $A = \min\limits_{x \in [a,b]} f(x)$ and $B = \max\limits_{x \in [a,b]} f(x)$

$\forall\, C : \quad A < C < B \quad \exists$ at least one value x_0

such that $\quad f(x_0) = C$.



Proof the intermediate value theorem.

Let's divide $[a,b]$ by $x_1 = \frac{a+b}{2}$:

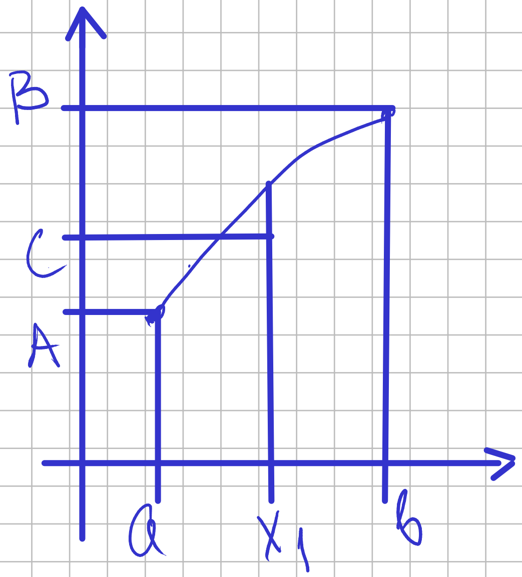
$$[a,b] = [a, x_1] \cup [x_1, b] \Rightarrow$$

1) $\min\limits_{x\in[a,x_1]} f(x) \leq C \leq \max\limits_{x\in[a,x_1]} f(x)$, or:

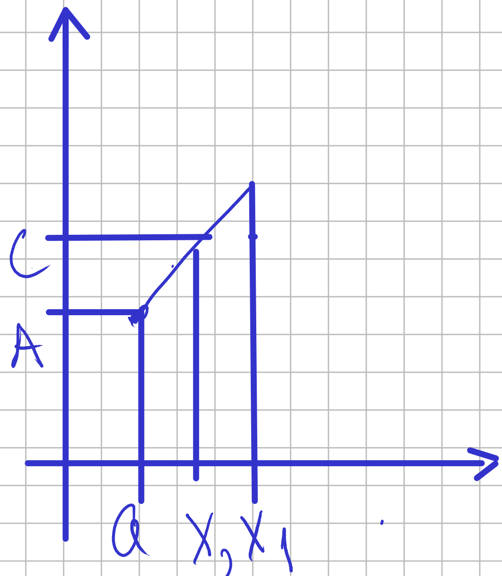
2) $\min\limits_{x\in[x_1,b]} f(x) \leq C \leq \max\limits_{x\in[x_1,b]} f(x)$, or

3) $f(x_1) = C$

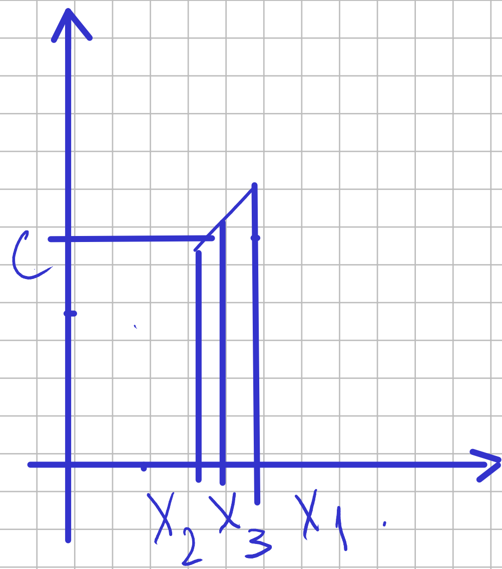
Repeate this step for 1) or 2) case.
We obtain convergent process to some x_0: $f(x_0)=C$.



$$X_1 = \frac{a+b}{2}$$



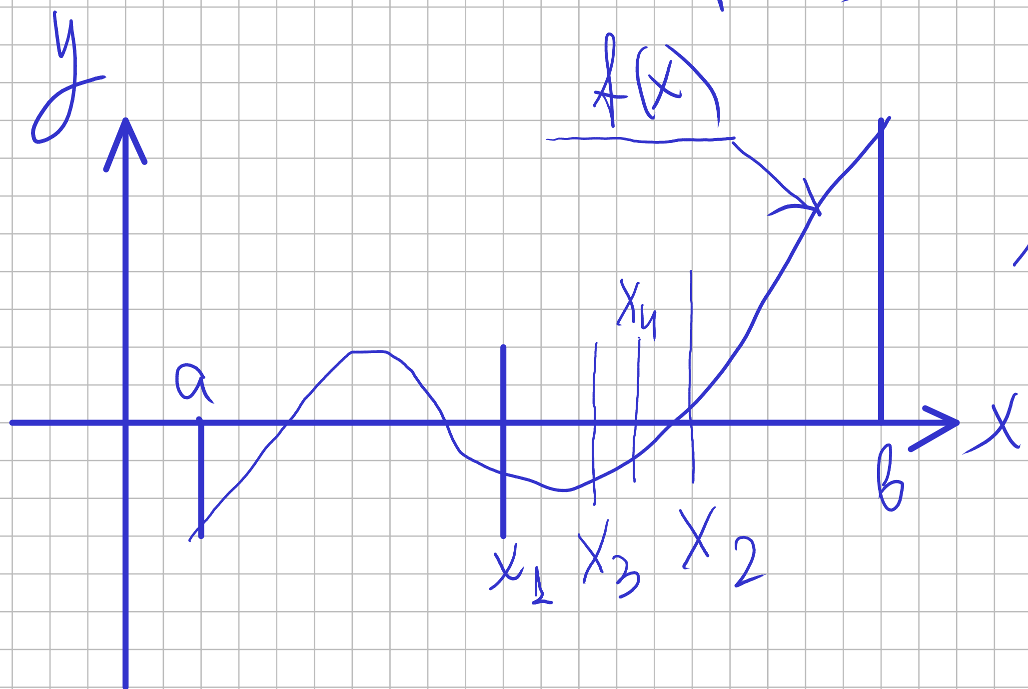
$$X_2 = \frac{a+X_1}{2}$$



$$X_3 = \frac{X_1+X_2}{2}$$

An application of the intermediate value theorem

Let $f(x)$ be continuous function on $[a,b]$ and $f(a) < 0 < f(b) \Rightarrow \exists\, x_0 \in [a,b]$ such that $f(x_0) = 0$.

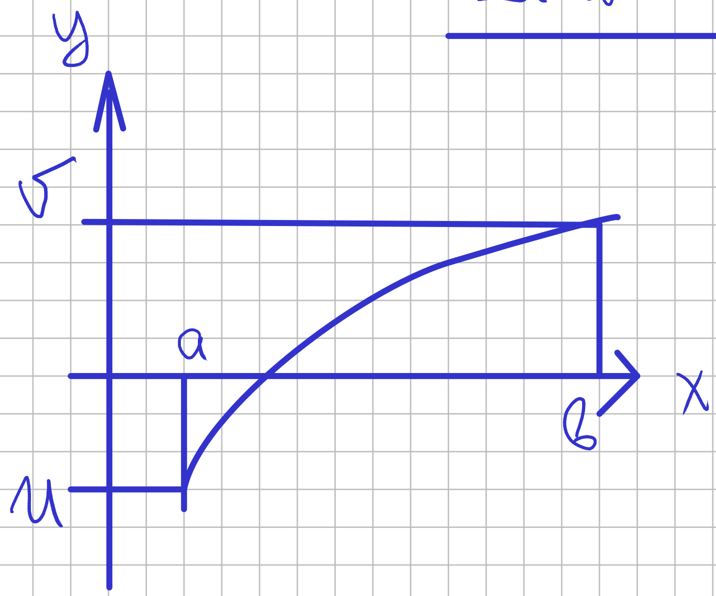


The bisection method for solving an equation

$$f(x) = 0$$

basics on the intermediate value theorem.

Inverse function.



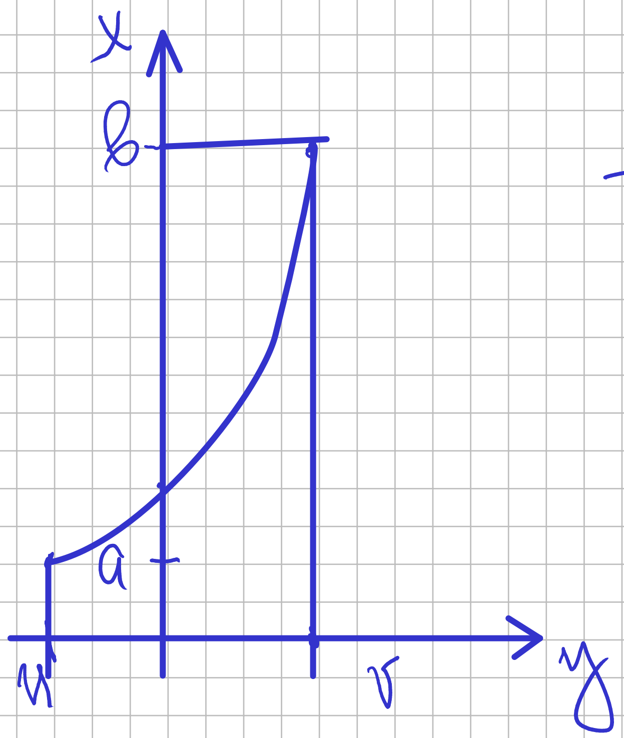
$f(x): x \in [a, b]$,

the range $f(x): [u, v]$

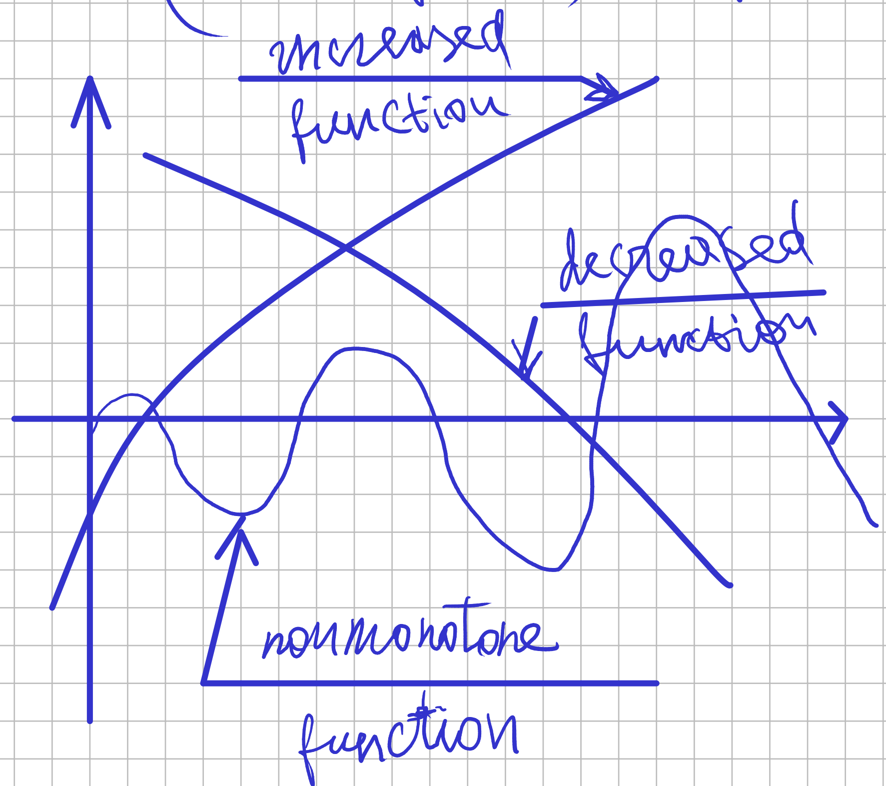
the inverse function $f^{-1}(y) = x$

The domain $f^{-1}(y): [u, v]$,

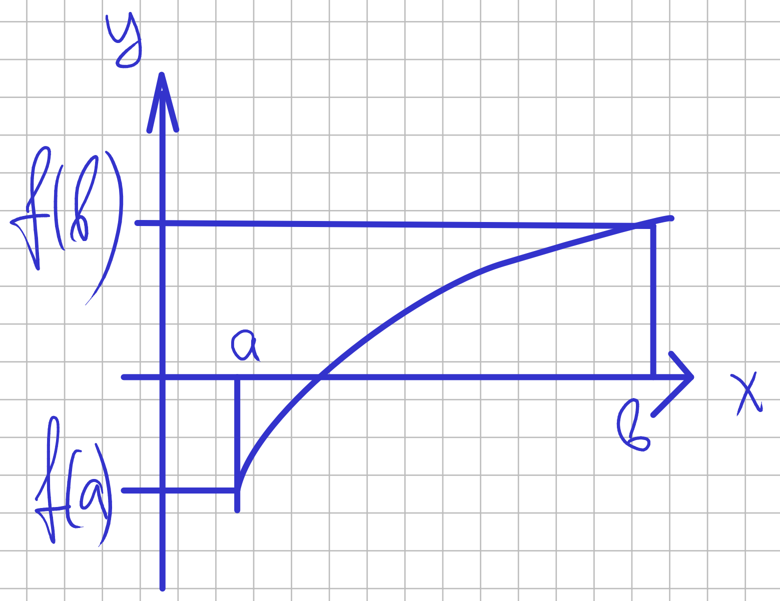
the range $f^{-1}(y): [a, b]$.

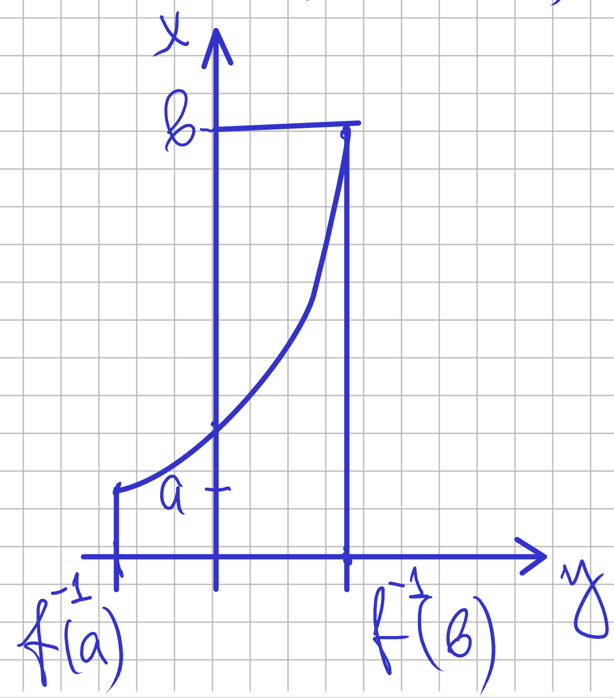


A function f, defined on numeric set E, is called mononously increased (decreased) if $\forall x_1, x_2 \in E$, and $x_1 < x_2$ $(x_1 > x_2)$ is true then $f(x_1) < f(x_2)$ $(\text{or } f(x_1) > f(x_2))$.



Theorem Let $f(x)$ be continuous and monotonously increasing on the interval $[a, b]$, then the inverse $f^{-1}(x)$ is one-valued monotonously increasing and continuous function on $[f(a), f(b)]$.





Let $f(a)=c$, $f(b)=d$.

1) The domain of $f^{-1}(x)$: $[c,d]$

$f(a)\leq f(x)\leq f(b) \Rightarrow \forall y\in[c,d]\ \exists x : f(x)=y$

2) The inverse function is one-valued.

Assume $\exists y\in[c,d] : f(x_1)=y,\ f(x_2)=y,\ x_1\neq x_2$

but we have two case

$x_1>x_2 \Rightarrow f(x_1)>f(x_2)$;

or

$x_1<x_2 \Rightarrow f(x_1)<f(x_2)$.

We obtain a contradiction!

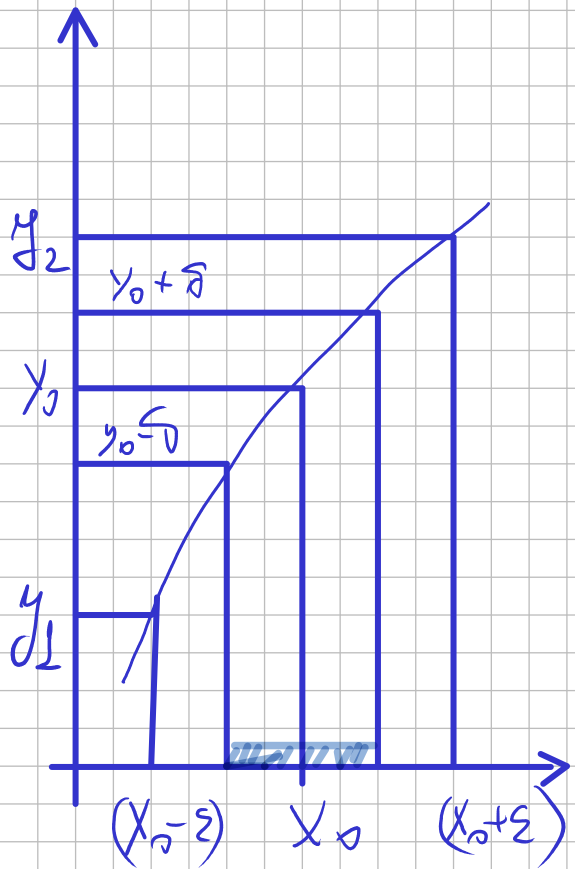
3) $f^{-1}(x)$ increases : $f^{-1}(y_1) < f^{-1}(y_2)$ if $y_1 < y_2$

Let $y_1 < y_2 \Rightarrow f^{-1}(y_1) = x_1$ and $f^{-1}(y_2) = x_2$

then there are 3 cases:

1. $x_1 = x_2 \Rightarrow y_1 = y_2$ - contradiction;
2. $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$ - contradiction;
3. $x_1 < x_2 \Rightarrow f(x_1) < f(x_2) : (y_1 < y_2)$!

4) $f^{-1}(y)$ is continuos function,



Assume $\exists\, y_0$: $c < y_0 < d$ and $f^{-1}(y_0) = x_0$

$\exists\, \varepsilon > 0$: $|x - x_0| < \varepsilon$,

$y_1 = f(x_0 - \varepsilon)$, $y_2 = f(x_0 + \varepsilon) \Rightarrow c < y_1 < y_0 < y_2 < d$

Let δ be such that $y_1 < y_0 - \delta$, $y_0 + \delta < y_2$

and $y_0 - \delta < y < y_0 + \delta \Rightarrow$

$x_0 - \varepsilon = f^{-1}(y_1) < f^{-1}(y) < f^{-1}(y_2) = x_0 + \varepsilon$.

Then: $\forall\, \varepsilon > 0\ \exists\, \delta(\varepsilon) > 0$: $y \in [y_0 - \delta, y_0 + \delta]$

$|f^{-1}(y) - f^{-1}(y_0)| < \varepsilon$.

Theorem Let $f(x)$ be continuous monotonously increasing on the interval (a, b), then the inverse $f^{-1}(x)$ is one-valued monotonously increasing and continuous function on $(f(a), f(b))$.

Properties of elementary functions

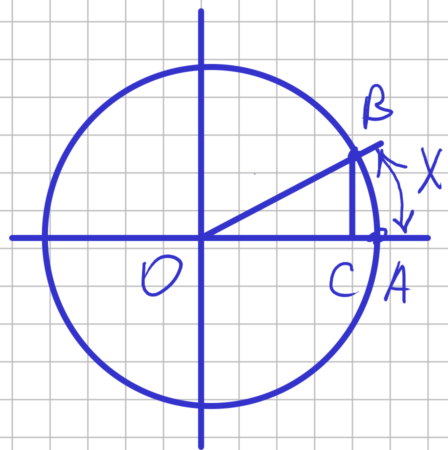
1) $\sin(x)$ is continuous function.

$$\sin(x+\varepsilon) - \sin(x) = \frac{1}{2}\cos\left(\frac{x+\varepsilon+x}{2}\right)\sin\left(\frac{x+\varepsilon-x}{2}\right) =$$

$$= \frac{1}{2}\cos\left(x+\frac{\varepsilon}{2}\right)\sin\left(\frac{\varepsilon}{2}\right)$$

$$|\sin(\alpha)| < |\alpha|$$

$\forall \varepsilon > 0 \;\; \exists \delta > 0 : \; |\sin(x+\varepsilon) - \sin(x)| < \varepsilon$



$l_{AB} = x$

$BC = \sin(x)$

2) x^n is continuos function

$$(x+\varepsilon)^2-x^2=(x+\varepsilon-x)(x+\varepsilon+x)=\varepsilon\cdot(2x+\varepsilon);$$

$$(x+\varepsilon)^3-x^3=(x+\varepsilon-x)\left((x+\varepsilon)^2+(x+\varepsilon)x+x^2\right)=$$

$$=\varepsilon\cdot\left((x+\varepsilon)^2+(x+\varepsilon)x+x^2\right);$$

$$(x+\varepsilon)^n-x^n=\left((x+\varepsilon)-x\right)\cdot\left((x+\varepsilon)^{n-1}+(x+\varepsilon)^{n-2}x+\dots+x^{n-1}\right)=$$

$$=\varepsilon\cdot\left((x+\varepsilon)^{n-1}+\dots+x^{n-1}\right)$$

$$a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\dots+ab^{n-2}+b^{n-1})$$

$$\lim_{x \to +0} (1+x)^{1/x} = e.$$

$\forall\ x < 1\ \exists\ n:\ \frac{1}{n+1} < x < \frac{1}{n}$ and $n+1 > \frac{1}{x} > n.$

$$\left(1+\frac{1}{n+1}\right)^{n} < (1+x)^{1/x} < \left(1+\frac{1}{n}\right)^{n+1}$$

$$\left(1+\frac{1}{n+1}\right)^{n+1} \cdot \left(1+\frac{1}{n+1}\right)^{-1} < (1+x)^{1/x} < \left(1+\frac{1}{n}\right)^{n} \cdot \left(1+\frac{1}{n}\right)$$

$n \to \infty$

$$e < \lim_{x \to +0} (1+x)^{1/x} < e$$

$$\lim_{x \to -0} (1+x)^{1/x} = \left| y = -x \right| = \lim_{y \to +0} (1-y)^{-1/y} =$$

$$\lim_{y \to +0} \left(\frac{1}{1-y} \right)^{1/y} = \lim_{y \to +0} \left(\frac{1-y+y}{1-y} \right)^{1/y} =$$

$$\lim_{y \to +0} \left(1 + \frac{y}{1-y} \right)^{1/y} = \left| z = \frac{y}{1-y}, \ \frac{1}{y} = \frac{1-y+y}{y} = \frac{1-y}{y} + 1 = \frac{1}{z} + 1 \right| =$$

$$\lim_{z \to +0} (1+z)^{\frac{1}{z}+1} = \lim_{z \to +0} (1+z)^{1/z} \cdot (1+z) = e$$

$$\lim_{x \to 0} \ln \left((1+x)^{1/x} \right) = \lim_{x \to 0} \frac{1}{x} \ln(1+x) = 1.$$

Asymptotic properties

If $\exists c \in \mathbb{R} : c|f(x)| \geq |g(x)| \quad \forall x \to x_0$

then we will write: $g(x) = O(f(x))$

and will say:

„g" is of order not exeed „f"

or : $g(x)$ equals big O of $f(x)$

another words:

$\forall \varepsilon > 0 \quad \exists c \in \mathbb{R} : c|f(x)| \geq g(x), \quad |x - x_0| < \varepsilon$.

Examples;

1) $\sin(x) = O(x)$, $x \to 0$;

2) $\ln(1+x) = O(x)$, $x \to 0$;

3) $(x + \sqrt{x}) = O(\sqrt{x})$, $x \to +0$,

4) $x \sin\left(\frac{1}{x}\right) = O(x)$, $x \to 0$

If $\frac{f(x)}{g(x)} \to 0$, $x \to x_0$, we will write:

$$f(x) = o(g(x)), \quad x \to x_0$$

and we will say:

„f is of order less than g"

briefly: „f" equals „o" small of g

Examples:

$$\sin(x) = o(1), \quad x \to 0$$

$$(x - x_0)^2 = o((x - x_0)) \quad x \to x_0$$

$$x - a = o(\sqrt{x-a}) \quad x \to a+0.$$

If $\dfrac{f(x)}{g(x)} = 1$, $x \to x_0$

we will write $f(x) \sim g(x)$, $x \to x_0$

and will say:

"f" is asymptotic to "g"

or $g(x)$ asymptotic approximates $f(x)$

examples: $x^2 + x \sim x$, $x \to 0$

$\sin(x) \sim x$, $x \to 0$

$x + \dfrac{1}{x} \sim \dfrac{1}{x}$, $x \to 0$.

Theorem. $f(x) \sim g(x)$, $x \to x_0$, $f(x_0) \neq 0$, $g(x_0) \neq 0$

is necessary and sufficient:

$$f(x) - g(x) = o(g(x)) \quad \text{or} \quad g(x) - f(x) = o(\cancel{f}(x)).$$

Proof. Let $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = 1 \Rightarrow \lim\limits_{x \to x_0} \frac{f(x)}{g(x)} - 1 = 0$

then $\lim\limits_{x \to x_0} \frac{f(x) - g(x)}{g(x)} = 0 \Rightarrow f(x) - g(x) = o(g(x))$

sufficiency: Let $g(x) - f(x) = o(g(x)) \Rightarrow$

$$1 - \frac{f(x)}{g(x)} = \frac{o(g(x))}{g(x)} \Rightarrow$$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$$

If $g \sim f$, $f \sim h \Rightarrow g \sim h$, $x \to x_0$

$$\lim_{x \to x_0} \frac{g(x)}{f(x)} \cdot \lim_{x \to x_0} \frac{f(x)}{h(x)} = \lim_{x \to x_0} \frac{g(x)}{f(x)} \cdot \frac{f(x)}{h(x)} = \lim_{x \to x_0} \frac{g(x)}{h(x)}$$

Usage of the asymptotic properties.

1) $$\lim_{x\to 0} \frac{3x+x^3}{2x+5x^2} = \lim_{x\to 0} \frac{x\cdot(3+x^2)}{x(2+5x)} = \lim_{x\to 0} \frac{3+x^2}{2+5x} = \frac{3}{2}$$

2) $$\lim_{x\to 1} \frac{\sin(\ln(x))}{x-1} = \lim_{y\to 0}\left(\frac{\sin(\ln(1+y))}{y}\right) =$$

$$= \lim_{y\to 0}\left(\frac{\sin(y+o(y))}{y}\right) = 1.$$