Applications of derivatives

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Previous lecture

Mean value theorems

L'Hospital's rule

Taylor's formula

High-order derivatives

We recall the formula for the *n*-th order derivative:

$$f^{(n)}(x) \equiv \frac{d^n f}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}f}{dx^{n-1}}\right).$$

Let's define a difference operator:

$$\begin{aligned} D(f(x)) &:= f(x+\Delta) - f(x), \\ D^2(f) &\equiv D(D(f(x))) = D(f(x+\Delta) - f(x)) \\ &= D(f(x+\Delta)) - D(f(x)) = \\ &\quad (f(x+2\Delta) - f(x+\Delta)) - (f(x+\Delta) - f(x)) = \\ &\quad f(x+2\Delta) - 2f(x+\Delta) + f(x). \end{aligned}$$

$$f''(x) \equiv \frac{d^2 f}{dx^2} = \lim_{\Delta\to 0}\frac{D^2(f(x))}{\Delta^2}.$$

High-order derivatives

Using the definition

$$D^n(f) \equiv D(D^{n-1}f),$$

the formula for the n-th derivative can be written as follows:

$$\frac{d^n f}{dx^n} = \lim_{\Delta\to 0} \frac{D^n(f(x))}{\Delta^n}.$$

High-order derivatives for parametric form of the functions

$$
\begin{aligned}
y &= y(t),\ \ x = x(t), \Rightarrow y(x) = y(t(x)),\\
\frac{dy}{dx} &= \frac{dt}{dx}\frac{d}{dt}\left(y(t)\right) = \frac{1}{\frac{dx}{dt}}\frac{d}{dt}\left(y(t)\right) = \frac{y'(t)}{x'(t)},\\
\frac{d^2y}{dx^2} &= \frac{dt}{dx}\frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right) = \frac{1}{x'(t)}\frac{y''(y)x'(t) - y'(t)x''(t))}{(x'(t))^2} = \\
& \frac{y''(y)x'(t) - y'(t)x''(t))}{(x'(t))^3}.
\end{aligned}
$$

High-order derivatives for parametric form of the functions

$$
\begin{aligned}
y &= y(t),\ x = x(t), \\
\frac{d^n y}{dx^n} &= \frac{dt}{dx}\frac{d}{dt}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) \equiv \frac{1}{\frac{dx}{dt}}\frac{d}{dt}\left(\frac{d^{n-1}y}{dx^{n-1}}\right).
\end{aligned}
$$

An example

$$y(t) = a\sin(t), \quad x(t) = b\cos(t), \quad t \in \mathbb{R}.$$

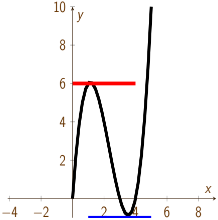
$$\frac{dy}{dx} = \frac{dt}{dx}\frac{dy}{dt} = \frac{1}{-b\sin(t)} a\cos(t) = -\frac{a\cos(t)}{b\sin(t)},$$

$$\frac{d^2y}{dx^2} = \frac{dt}{dx}\frac{d}{dt}\frac{dy}{dx} = \frac{-1}{b\sin(t)}\frac{d}{dt}\left(-\frac{a\cos(t)}{b\sin(t)}\right) =$$

$$\frac{-1}{b\sin(t)}\frac{a\sin(t)\sin(t) + a\cos(t)\cos(t)}{b\sin^2(t)} =$$

$$-\frac{a}{b^2\sin^3(t)}.$$

Fermat's lemma



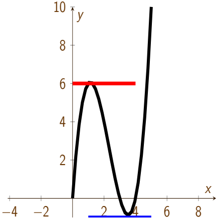
Let $f(x)$ is defined $x \in (a, b)$
and $\exists \xi \in (a, b) : \max_{x \in (a,b)} f(x) = f(\xi)$
(or $\min_{x \in (a,b)} f(x) = f(\xi)$),
if $\exists f'(\xi) \to f'(\xi) = 0$.

Proof Let $f(\xi)$ be a maximum:

$$x < \xi \Rightarrow \frac{f(x) - f(\xi)}{x - \xi} \geq 0,$$

$$x > \xi \Rightarrow \frac{f(x) - f(\xi)}{x - \xi} \leq 0.$$

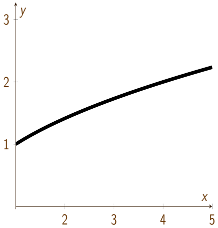
Fermat's lemma



$$f'(\xi) = \lim_{x \to \xi} \frac{f(x) - f(\xi)}{x - \xi},$$

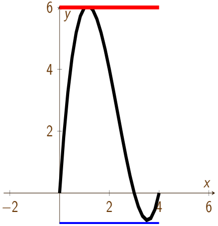
$$\left.\begin{array}{l} x < \xi \Rightarrow f'(\xi) \geq 0, \\ x > \xi \Rightarrow f'(\xi) \leq 0, \end{array}\right\} \Rightarrow f'(\xi) = 0.$$

Fermat's lemma. A counterexample



If $x \in [a, b]$ then a monotonous increased function has a minimum at $\xi = a$ and maximum at $x = b$ but if $\exists f'(a),\ f(b)$, then $f'(a) > 0$ and $f'(b) > 0$

Rolle's theorem

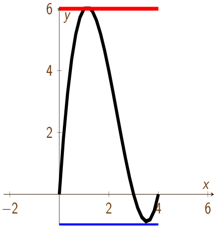


Let $f(x)$ be such that:

- continuous at $x \in [a, b]$,
- differentiable at $x \in (a, b)$,
- $f(a) = f(b)$.

then $\exists \xi \in (a, b) : f'(\xi) = 0$.

Rolle's theorem. A proof.

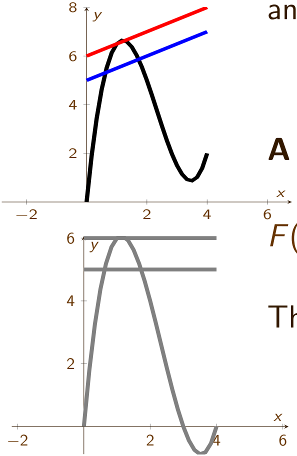


Let's define

$$m = \min_{x \in (a,b)} f(x), \quad M = \max_{x \in (a,b)} f(x).$$

If $m = M$, then $f(x) = C, \Rightarrow f'(x) \equiv 0$.
If $m < M$, and $f(a) = f(b)$, hence $f(x) = M$ (or $f(x) = m$): $x \notin \{a, b\}$.
Then, using the Fermat's lemma, one gets $\exists \xi \in (a, b) : f'(\xi) = 0$.

Lagrange's theorem



Let $f(x)$ be continuous on $[a, b]$ and $\exists f'(x), x \in (a, b)$, then $\exists \xi \in (a, b)$:

$$f(b) - f(a) = f'(\xi)(b - a).$$

A proof. Le's consider

$$F(x) = (f(x) - f(a)) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then

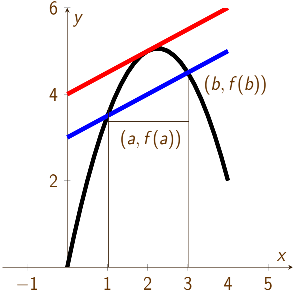
$$F(a) = F(b) = 0.$$

Lagrange's theorem. The proof.

So, using Rolle's theorem, one obtains

$$\begin{aligned} \exists \xi \in (a,b): \;\; F'(\xi) = 0, \\ f'(\xi) - \frac{f(b)-f(a)}{b-a} = 0, \\ f'(\xi) = \frac{f(b)-f(a)}{b-a}. \end{aligned}$$

A geometrical sense of the Lagrange's theorem



The formula

$$f'(\xi) = \frac{f(b) - f(a)}{b - a},$$
$$\frac{f(b) - f(a)}{b - a} = \tan(\alpha),$$
$$\exists f'(\xi) = \tan(\alpha).$$

means that there exists a tangent line which is parallel to the chord a the points $(a, f(a)), (b, f(b))$.

Usage of the Lagrange's theorem

$$f(b) = f(a) + f'(\xi)(b-a), \quad \xi \in (a,b).$$

For small values of $(b-a)$:

$$f(b) \sim f(a) + f'(a)(b-a).$$

Let's consider $f(x) = \sqrt{x}$, then $f'(x) = \frac{1}{2\sqrt{x}}$. Then:

$$\sqrt{b} = \sqrt{a} + \frac{1}{2\sqrt{a}}(b-a),$$

$$a = 81, \quad b = 90;$$

$$\sqrt{90} \sim 9 + \frac{1}{18}9 = 9.5 \quad \sqrt{90} \sim 9.4868$$

Cauchy's theorem

Let $f(x)$ and $g(x)$ be such that:

- continuous on $[a, b]$;
- $\exists f'(x), g'(x), x \in (a, b)$;
- $g'(x) \neq 0, \quad x \in (x, b)$.

Then there exists $\xi \in (a, b)$:

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

Cauchy's theorem. A proof.

Let's consider:

$$F(x) = (f(x) - f(a)) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)),$$
$$F(a) = 0, \quad F(b) = 0.$$

Due to the Rolle's theorem $\exists \xi \in (a, b)$:

$$F'(\xi) = 0, \quad f'(\xi) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(\xi) = 0,$$
$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

An exercise from Kudryavtsev handbook

Let's consider $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ and $f(0) = 0$. Then $f(x)$ is defined on $[0, x]$

$$x^2 \sin\left(\frac{1}{x}\right) = \left(2\xi \sin\left(\frac{1}{\xi}\right) - \cos\left(\frac{1}{\xi}\right)\right) x,$$

$$x \sin\left(\frac{1}{x}\right) = 2\xi \sin\left(\frac{1}{\xi}\right) - \cos\left(\frac{1}{\xi}\right),$$

$$x \to 0, \Rightarrow \xi \to 0 :$$

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = \lim_{\xi\to 0} 2\xi \sin\left(\frac{1}{\xi}\right) - \lim_{\xi\to 0} \cos\left(\frac{1}{\xi}\right).$$

$$0 = 0 - \lim_{\xi\to 0} \cos\left(\frac{1}{\xi}\right).$$

L'Hospital's rule

If $\exists f'(x_0), g'(x_0)$ and $f(x_0) = 0$, $g(x_0) = 0$, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$$

Proof:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \lim_{x \to x_0} \frac{f'(\xi)}{g'(\xi)} = \lim_{\xi \to x_0} \frac{f'(\xi)}{g'(\xi)}$$

L'Hospital's rule. Corollaries

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = |x = 1/t| = \lim_{t\to 0}\frac{f(1/t)}{g(1/t)} =$$

$$\lim_{t\to 0}\frac{\frac{-1}{t^2}f'(1/t)}{\frac{-1}{t^2}g'(1/t)} = \lim_{t\to 0}\frac{f'(1/t)}{g'(1/t)} =$$

$$\lim_{x\to\infty}\frac{f'(x)}{g'(x)}.$$

L'Hospital's rule. Corollaries

Let $f(x)$ and $g(x)$ be such that $f(x) \to \infty, x \to x_0$ and $g(x) \to \infty, x \to x_0$. Take a: $a \neq x_0$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f(x)}{g(x)} \frac{(1 - \frac{f(a)}{f(x)})}{(1 - \frac{g(a)}{g(x)})} =$$

$$\lim_{x \to x_0} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{\xi \to x_0} \frac{f'(\xi)}{g'(\xi)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

L'Hospital's rule. Examples

$$\lim_{x\to a} \frac{a^x - x^a}{x - a} = \lim_{x\to a}(a^x \log(a) - ax^{a-1}) = (log(a) - 1)a^a.$$

$$\lim_{x\to\infty} \frac{\log(x)}{x^a} = \lim_{x\to\infty} \frac{1/x}{ax^{a-1}} = \frac{1}{a} \lim_{x\to\infty} \frac{1}{x^a} = 0.$$

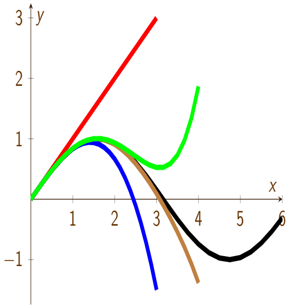
Counterexample:

$$\lim_{x\to\infty} \frac{x - \sin(x)}{x + \sin(x)} = \lim_{x\to\infty} \frac{1 - \cos(x)}{1 + \cos(x)}$$

The right-hand side limit does not exists, but:

$$\lim_{x\to\infty} \frac{x - \sin(x)}{x + \sin(x)} = 1.$$

Taylor's formula



Main idea of the Taylor formula is an approximation of given curve by polynomials of different orders. Here one can see the $\sin(x)$ (in black) and the approximation by polynomials of first (red), 3-th (blue), 5-th order (green) and 7-th orders.

Taylor's formula

Let $f(x)$ be a function which has n derivatives at $x = x_0$.

$$f(x) = f(x_0) + f'(\xi)(x - x_0) + k_2(x - x_0)^2 + k_3(x - x_0)^3 + \cdots + k_n(x - x_0)^n + o((x - x_0)^n).$$

$$f''(x) = 2k_2 + 3 \cdot 2 \cdot k_3(x - x_0) + \cdots + o((x - x_0)^{n-2}),$$

$$k_2 = \frac{f''(x_0)}{2},$$

$$f'''(x) = 3 \cdot 2 \cdot 1 \cdot k_3 + \cdots + o((x - x_0)^{n-2}),$$

$$k_3 = \frac{f'''(x_0)}{3 \cdot 2},$$

$$\ddots,$$

$$k_n = \frac{f^{(n)}(x_0)}{n!}.$$

Taylor's formula. Peano form of the remainder.

Theorem.

Let $f(x)$ has derivatives of n-th order at $x = x_0$, then:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o((x - x_0)^n).$$

Taylor's formula. Lagrange form of the remainder.

Let's consider

$$\phi(x) = f(x) - \left(f(x_0) + \sum_{k=1}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \right),$$

$$\phi(x) = \frac{f^{(n)}(\xi)}{n!}(x - x_0)^n.$$

Then

$$f(x) = f(x_0) + \sum_{k=1}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n)}(\xi)}{k!}(x - x_0)^n.$$

The Taylor formula. Examples

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5).$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3).$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + o(x^3).$$

Summary

Previous lecture

Mean value theorems

L'Hospital's rule

Taylor's formula