Antiderivatives

O.M. Kiselev
o.kiselev@innopolis.ru

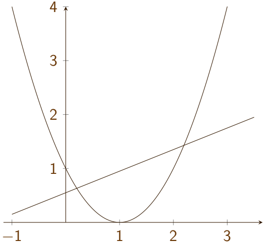
Innopolis university

October 28, 2022

A convexity and a concavity

The second derivative of the tangent line is equal to zero. Then the function with positive second derivative grows faster with respect to the tangent line. Then the tangent line lower than the curve at right side of the touch point and vice wise the negative second derivative means the function grows slowly with respect to tangent line and curve lower that the tangent line in the right side with respect to the touch point.

A convex curve



Let's consider a curve $f(x)$ on a segment $x \in [a, b]$.

Theorem about a convex curve

If the segment of the curve lies lower than any secant line then the curve is convex on this interval. If this curve has a second derivative, then the second derivative is positive.

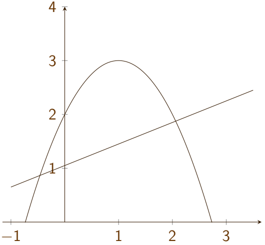
Prof of the theorem about a convex curve

$$y(x) - f(x) = \frac{(f'(\xi_2) - f'(\xi_1))(x_2 - x)(x - x_1)}{x_2 - x_1} =$$
$$\frac{f''(\eta)(\xi_2 - \xi_1)(x_2 - x)(x - x_1)}{x_2 - x_1},$$
$$x_1 < \xi_1 < x < \xi_2 < x_2, \quad \xi_1 < \eta < \xi_2.$$

Then

$$y(x) - f(x) > 0 \Rightarrow f''(\eta) > 0.$$

A concave function



Theorem about a concave curve

If the segment of the curve lies upper than any secant line then the curve is concave function on this interval. If this curve has a second derivative, then the second derivative is negative.

A concave function

Proof

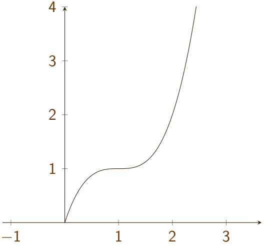
is follows from the equality:

$$y(x) - f(x) = \frac{f''(\eta)(\xi_2 - \xi_1)(x_2 - x)(x - x_1)}{x_2 - x_1},$$
$$x_1 < \xi_1 < x < \xi_2 < x_2, \quad \xi_1 < \eta < \xi_2.$$

Then

$$y(x) - f(x) < 0 \Rightarrow f''(\eta) < 0.$$

A point of inflection



If $f''(x_0) = 0$ and $f''(x_1)f''(x_2) < 0$, in small neighborhood of x_0, $x_1 < x_0 < x_2$ then x_0 is an inflection point.

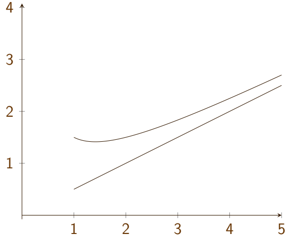
Theorem about the inflection point

If $f''(x_0) = 0$ and $f'''(x_0) \neq 0$ then x_0 is an inflection point.

Proof

$$f''(x) - f''(x_0) \sim f'''(x_0)(x - x_0).$$

Asymptotes



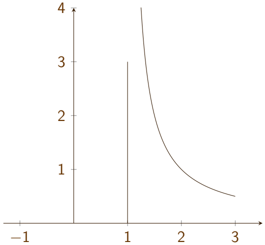
Definition

Let $f(x)$ be defined $\forall x > a$. If $\exists k, l \in \mathbb{R}$:

$$f(x) - kx - l = o(1), \quad x \to \infty,$$

then the straight line $kx + l$ is called asymptote

Construction of the asymptote



If asymptote exists then the parameters of the asymptote are

$$k = \lim_{x \to \infty} \frac{f(x)}{x},$$

$$l = \lim_{x \to \infty} f(x) - kx.$$

If

$$\lim_{x \to x_0} f(x) = \infty,$$

then the $x = x_0$ is called a vertical asymptote.

A table of derivatives of elementary functions

$$
\begin{aligned}
\frac{d}{dx} x^n &= n x^{n-1}, \\
\frac{d}{dx} e^x &= e^x, \\
\frac{d}{dx} \log(x) &= \frac{1}{x}, \\
\frac{d}{dx} \sin(x) &= \cos(x), \\
\frac{d}{dx} \cos(x) &= -\sin(x), \\
\frac{d}{dx} \tan(x) &= \frac{1}{\cos^2(x)}, \\
\frac{d}{dx} \arcsin(x) &= \frac{1}{\sqrt{1-x^2}}, \\
\frac{d}{dx} \arctan(x) &= \frac{1}{1+x^2}.
\end{aligned}
$$

A definition of an antiderivative

Definition.

Let a function $f(x)$ be defined at some interval E. The function $F(x)$ such that

$$\frac{d}{dx}F(x) = f(x), \quad \forall x \in E.$$

is called the antiderivative of the function $f(x)$ on E.

If $F(x)$ is antiderivative of $f(x)$, then $F(x) + C$ is antiderivative of $f(x)$ also. **Proof.**

$$\frac{d}{dx}(F(x) + C) = f(x).$$

Applications in physics

Suppose we know the velocity of the straightforward movement $v(t)$. Define the instant value of a coordinate as $x(t)$.
Then

$$\dot{x} = v.$$

Hence the $x(t)$ is an antiderivative of the velocity $v(t)$. If we know the acceleration $a(t)$ then the velocity $v(t)$ is an antiderivative of $a(t)$

An indefinite integral

Let's rewrite the formula for an antiderivative in differentials:

$$dF(x) = f(x)dx.$$

Definition

$$\int f(x)dx \equiv F(x) + C, \quad \forall C \in \mathbb{R}$$

the left-hand side of the formula is called indefinite integral.

Definition of an indefinite integral

$$\int f(x)dx \equiv F(x) + C, \quad \forall C \in \mathbb{R}$$

- $\int$ is an integral sign.
- $f(x)$ is called an integrand.
- dx is a differential of the variable of integration.
- $F(x)$ is an antiderivative of the integrand.
- C is an arbitrary constant.

Properties of the indefinite integrals

- Let $F'(x) = f(x)$ then
  $\int f(x)dx = \int dF(x) = F(x) + C, \quad C \in \mathbb{R}.$
- $d\left(\int f(x)dx\right) = F(x).$
- If $F'(x) = f(x)$ and $G'(x) = g(x)$ on $x \in E$, then

$$\int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx = F(x)+G(x)+C.$$

- If $F'(x) = f(x)$ then

$$\int \lambda f(x)dx = \lambda \int f(x)dx = \lambda F(x) + C, \forall \lambda \in \mathbb{R}.$$

A table of antiderivatives for elementary functions

$$
\begin{aligned}
\int x^{n-1} dx &= \frac{x^n}{n} + C, \\
\int e^x &= e^x + C, \\
\int \frac{dx}{x} &= \log(|x|) + C, \\
\int \cos(x) dx &= \sin(x) + C, \\
\int \sin(x) dx &= -\cos(x) + C, \\
\int \frac{dx}{\cos^2(x)} &= \tan(x) + C, \\
\int \frac{dx}{\sqrt{1-x^2}} &= \arcsin(x) + C, \\
\int \frac{dx}{1+x^2} &= \arctan(x) + C.
\end{aligned}
$$

Applications in physics

Consider a free falling load with the vertical acceleration $a = -g = \text{const}$ then the velocity as the indefinite integral:

$$v(t) = -gt + C$$

If we know the initial velocity $v(0) = v_0$, then

$$v(0) = C = v_0,$$
$$v(t) = v_0 - gt.$$

Applications in physics

The instant coordinate can be considered as an indefinite integral of the velocity:

$$x(t) = v_0 t - g\frac{t^2}{2} + C.$$

If we know the initial position x_0, then

$$x(0) = C = x_0.$$

Finally one gets the law for a free falling load:

$$x(t) = x_0 + v_0 t - g\frac{t^2}{2}.$$

Substitutions

Let's consider the chain rule for the derivative:

$$\frac{d}{dx}F(G(x)) = F'(G(x))G'(x),$$

then in the differential form:

$$d(F(G(x))) = F'(G(x))G'(x)dx = F'(G(x))d(G(x)).$$

Let's define $y = G(x)$, then:

$$F'(G(x))d(G(x) = F'(y)dy.$$

It yields:

$$\int F'(G(x))G'(x)dx = \int F'(G(x))dG(x) = \int F'(y)dy,$$

Substitutions

$$\int F'(G(x))G'(x)dx = \int F'(G(x))dG(x) = \int F'(y)dy,$$

then:

$$\int dF(y) = F(y) + C = F(G(x)) + C.$$

Examples

$$\int e^{\lambda x} dx = \int e^{\lambda x} \frac{d(\lambda x)}{\lambda} =$$

$$\frac{1}{\lambda} \int e^{y} dy = \frac{1}{\lambda} e^{y} + C = \frac{e^{\lambda x}}{\lambda} + C.$$

$$\int cos(x^2) x dx = \int cos(x^2) \frac{d(x^2)}{2} = \frac{1}{2} \int \cos(y) dy =$$

$$\frac{1}{2} \sin(y) + C = \frac{1}{2} \sin(x^2) + C.$$

Theorem about integration by substitution

Theorem about integration by substitution

Let $F(x)$ and $G(x)$ are differentiable functions and range of $G(x)$ in domain of the $F(x)$. Then

$$\int F'(G(x))G'(x)dx = F(G(x)) + C.$$

Proof. Differentiate this formula straightforward!

Applications in population dynamics

Let's assume that a population of fishes in a pond increases proportional to the numbers of the fishes and initial quantity of the fish was n_0 This means:

$$\frac{d}{dt}n = kn,$$

or in differential form:

$$dn = kndt.$$

Applications in population dynamics

$$\frac{dn}{n} = kdt.$$

Then the antiderivative for both parts give as the dependency

$$\log|n| = kt + c, \quad c \equiv \log|C| \in \mathbb{R}.$$

Then

$$n(t) = Ce^{kt},$$

and due to initial value of n_0 ones gets:

$$n(t) = n_0 e^{kt}.$$

An integration by parts

Let's consider:

$$\frac{d}{dx}(u(x)v(x)) = \frac{du(x)}{dx}v(x) + u(x)\frac{dv(x)}{dx}.$$

These formula in the differential form can be represented as:

$$d(uv) = d(u(x))v(x) + u(x)d(v(x)),$$

or the same:

$$u(x)dv(x) = d(uv) - d(u(x))v(x).$$

This yields

$$\int u(x)v'(x)dx = \int d(uv) - \int v(x)u'(x)dx,$$

or

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx.$$

An integration by parts

Theorem about integration by parts

If the function $u(x)$ and $v(x)$ are differentiable on some interval E, then on the interval the following formula is valid:

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx.$$

Proof. Differentiate the formula!

Example

$$\int x\log(x)dx = \frac{1}{2}\int \log(x)d(x^2) =$$
$$\frac{1}{2}\log(x)x^2 - \frac{1}{2}\int x^2\frac{dx}{x} =$$
$$\frac{x^2}{2}\log(x) - \frac{1}{2}\int xdx =$$
$$\frac{x^2}{2}\log(x) - \frac{1}{4}x^2 + C.$$

An example

$$\int e^x \sin(x)dx = e^x \sin(x) - \int e^x \cos(x)dx =$$

$$e^x \sin(x) - e^x \cos(x) - \int e^x \sin(x)dx.$$

$$2\int e^x \sin(x)dx = e^x \sin(x) - e^x \cos(x) + C,$$

$$\int e^x \sin(x)dx = \frac{e^x \sin(x) - e^x \cos(x)}{2} + C.$$

Summary

Previous lecture

Asymptotes

Antiderivatives

Integration techniques